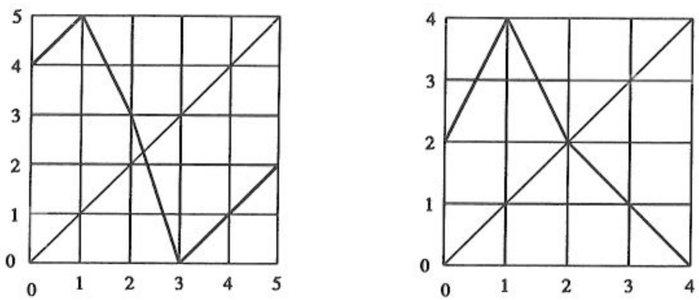

(a) (b)

Figure 11.4: Two graphs with no odd periods.

By the way, there's a nice pattern here that deserves mention.

Period 5, but not period 3:

$$1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1. \quad (11.7)$$

Period 7, but not period 5:

$$1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1. \quad (11.8)$$

See how (11.8) is obtained from (11.7)? Just add one to each point in the period 5 orbit and then add the iteration $7 \mapsto 1$ at the end. Similarly, we may construct a map with a period 9 orbit, but not period 7. The necessary 9-cycle would be

$$1 \mapsto 5 \mapsto 6 \mapsto 4 \mapsto 7 \mapsto 3 \mapsto 8 \mapsto 2 \mapsto 9 \mapsto 1. \quad (11.9)$$

The map corresponding to (11.9) is not hard to construct, and is left as an exercise.

7. Consider the graph in Figure 11.4a. Prove that this function has a cycle of period 6 but no cycles of any odd period.

This function has the 6-cycle

$$0 \mapsto 4 \mapsto 1 \mapsto 5 \mapsto 2 \mapsto 3 \mapsto 0$$

which may be checked easily from the graph, and so F has periodic points of all even periods. The mappings

$$[0,1] \mapsto [4,5] \mapsto [1,2] \mapsto [3,5] \mapsto [0,2] \mapsto [3,5] \mapsto \cdots$$

$$[2,3] \mapsto [0,2] \mapsto [0,5] \mapsto \cdots$$

clearly show the absence of odd periodic points, however. Take period 3 points, for instance. Since

$$F^3[0,1] = [3,5] \quad (11.10)$$

$$F^3[1,2] = [3,5] \quad (11.11)$$

$$F^3[3,5] = [0,2], \quad (11.12)$$

there are no period 3 points in these intervals since $[0,2]$ maps to $[3,5]$, and vice versa. The only remaining interval is $[2,3]$, but it contains absolutely *no* periodic points save the lone fixed point of F. So F has no period 3 points.

Let's also check period 5 points. Again, there aren't any since (11.10–11.12) hold with F^3 replaced with F^5. In fact, there are no periodic points with *any* odd period since

$$F^{2n+1}[0,1] = [3,5]$$

$$F^{2n+1}[1,2] = [3,5]$$

$$F^{2n+1}[3,5] = [0,2]$$

for all positive integers n.

Here's a totally different approach to this problem: First, verify that the piecewise linear function in Figure 11.5 has the period 3 orbit

$$0 \mapsto 1/2 \mapsto 1 \mapsto 0.$$

Since this function is continuous, Sarkovskii's theorem guarantees that it has periodic points of *all* periods.