So every point in $[0,1] \cup [2,3]$ is of period 4. Now $G[1,2] = [0,2]$ and $G[0,2] = [0,3]$. But as soon as a point leaves the closed interval $[1,2]$, it's locked into a 4-cycle. So the question is: are there points in $[1,2]$ that remain in $[1,2]$ for all time? The answer is no! The orbits of points close to the fixed point $x = 4/3$ oscillate away from the fixed point (since it's repelling) and eventually enter $[0,1]$, an interval of 4-cycles.

In summary, G has a repelling fixed point, a neutral 2-cycle, and a whole bunch of neutral period 4 points. Everything else is eventually periodic with period 4. We remark that G is the **double** of $x \mapsto 3 - x$ on $[0,3]$ which is easily seen to have nothing but period 2 points (see Devaney's *An Introduction to Chaotic Dynamical Systems*, Second Edition, pp. 67–68).

Now what about the dynamics of F? From the graph in Figure 11.2a, it's clear that

$$F[0,1] = [1,2] \tag{11.3}$$

$$F[1,2] = [2,3] \tag{11.4}$$

$$F[2,3] = [0,3] \tag{11.5}$$

and so we have $[0,1] \subset [0,3] = F^3[0,1]$. Therefore, F has a period 3 point in $[0,1]$ and this 3-cycle must be repelling. (How do we know it's repelling? Because the mappings (11.3–11.5) have constant slope 1, 1, and -3, respectively, and hence, the derivative of the period 3 point is $1 \cdot 1 \cdot (-3) = -3$.)

6. Consider the piecewise linear graph in Fig. 11.3. Prove that this function has a cycle of period 7 but not period 5.

Assuming $F\colon [1,7] \to [1,7]$, the 7-cycle

$$1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1$$

is repelling since each lap of F in Fig. 11.3 has constant slope greater than or equal to one in absolute value. Now, let's apply F five times to each of the unit intervals in $[1,7]$ and see what happens:

- $[1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7] \mapsto [1,6] \mapsto [2,7]$ and $[1,2] \cap [2,7] = \{3\} \in \mathrm{per}_7 F$.
- $[2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7]$ and $[2,3] \cap [3,7] = \{3\} \in \mathrm{per}_7 F$.

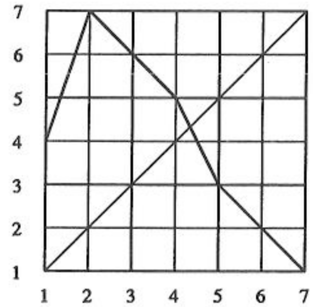

Figure 11.3: This function has a 7-cycle but no 5-cycle.

- $[3,4] \mapsto [5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7]$ and $[3,4] \cap [4,7] = \{4\} \in \mathrm{per}_7 F$.
- $[5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5]$ and $[5,6] \cap [1,5] = \{5\} \in \mathrm{per}_7 F$.
- $[6,7] \mapsto [1,2] \mapsto [4,7] \mapsto [1,5] \mapsto [3,7] \mapsto [1,6]$ and $[6,7] \cap [1,6] = \{6\} \in \mathrm{per}_7 F$.

Thus, there are no period 5 points in these intervals. Also notice that

$$[3,4] \mapsto [5,6] \mapsto [2,3] \mapsto [6,7] \mapsto [1,2] \mapsto \cdots \mapsto [1,7].$$

But what about $[4,5]$? Since

$$[4,5] \mapsto [3,5] \mapsto [3,6] \mapsto [2,6] \mapsto [2,7] \mapsto [1,7] \tag{11.6}$$

there is indeed a period 5 point in $[4,5]$. But we claim that there's exactly *one* such point in $[4,5]$, and in fact, it's the fixed point $x = 13/3$. This is because each of the mappings in (11.6) is strictly decreasing and therefore $F^5\colon [4,5] \to [1,7]$ is strictly decreasing (since the composition of an odd number of decreasing functions is decreasing). Thus F^5 has a unique fixed point in $[4,5]$, and moreover, this point must be the fixed point of F.

Does F have a 3-cycle? No, for if it did, it would also have a 5-cycle which we've already shown does not exist.