In either case, $D(x) = 0.b_2b_3b_4\ldots$ and so we see that doubling is equivalent to the shift. Consequently, anything true (in the dynamical sense) of the shift is also true of doubling, and in particular, D is chaotic because σ is.

21. Prove that the function

$$T(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2-2x & \text{if } x > 1/2 \end{cases}$$

is chaotic on $[0,1]$.

We will show that the doubling map is semi-conjugate to the tent map T via T itself! That is, we will show that

$$\begin{array}{ccc} [0,1] & \xrightarrow{D} & [0,1] \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ [0,1] & \xrightarrow{T} & [0,1] \end{array}$$

commutes. (Note that the doubling map D needs to be defined on the *closed* unit interval for this to work.) Thus T will be chaotic by virtue of Exercise 20. This is because orbits under iteration of D map to dynamically equivalent orbits under T. In fact, we now prove by induction that

$$T \circ D^{n-1} = T^n \qquad (10.1)$$

for all $n > 0$. Suppose Equation 10.1 is true for $n := k$. Then

$$\begin{aligned} &T \circ D^{k-1} = T^k \\ \Rightarrow\ &T \circ T \circ D^{k-1} = T \circ T^k, \end{aligned}$$

and since $T \circ D = T \circ T$ (this will be verified in a moment) we have

$$T \circ D^k = T^{k+1}$$

which completes the inductive proof. We remark that (10.1) gives an explicit formula for $T^n(x)$ since we already know that $D^{n-1}(x) = 2^{n-1}x \bmod 1$.

We now show that D is semi-conjugate to T via T, or in other words, that $T \circ D = T \circ T$. There are four cases to consider for $T \circ T$:

$$\begin{array}{lclcl} 0 \leq x \leq 1/4 & \Rightarrow & 0 \leq T(x) \leq 1/2 & \Rightarrow & T \circ T(x) = T(2x) \\ &&&& = 2(2x) \\ &&&& = 4x \\ 1/4 \leq x \leq 1/2 & \Rightarrow & 1/2 \leq T(x) \leq 1 & \Rightarrow & T \circ T(x) = T(2x) \\ &&&& = 2 - 2(2x) \\ &&&& = 2 - 4x \\ 1/2 \leq x \leq 3/4 & \Rightarrow & 1/2 \leq T(x) \leq 1 & \Rightarrow & T \circ T(x) = T(2-2x) \\ &&&& = 2 - 2(2-2x) \\ &&&& = 4x - 2 \\ 3/4 \leq x \leq 1 & \Rightarrow & 0 \leq T(x) \leq 1/2 & \Rightarrow & T \circ T(x) = T(2-2x) \\ &&&& = 2(2-2x) \\ &&&& = 4 - 4x \end{array}$$

Similarly, there are four cases for $T \circ D$:

$$\begin{array}{lclcl} 0 \leq x \leq 1/4 & \Rightarrow & 0 \leq D(x) \leq 1/2 & \Rightarrow & T \circ D(x) = T(2x) \\ &&&& = 2(2x) \\ &&&& = 4x \\ 1/4 \leq x < 1/2 & \Rightarrow & 1/2 \leq D(x) < 1 & \Rightarrow & T \circ D(x) = T(2x) \\ &&&& = 2 - 2(2x) \\ &&&& = 2 - 4x \\ 1/2 \leq x \leq 3/4 & \Rightarrow & 0 \leq D(x) \leq 1/2 & \Rightarrow & T \circ D(x) = T(2x-1) \\ &&&& = 2(2x-1) \\ &&&& = 4x - 2 \\ 3/4 \leq x < 1 & \Rightarrow & 1/2 \leq D(x) < 1 & \Rightarrow & T \circ D(x) = T(2x-1) \\ &&&& = 2 - 2(2x-1) \\ &&&& = 4 - 4x \end{array}$$

We have to be a little bit careful at $x = 1/2$ since D is not continuous there, and also at $x = 1$ since we haven't yet defined $D(1)$. But the reader may check that $T \circ D(1/2) = T \circ T(1/2) = 0$, and that $T \circ D(1) = T \circ T(1) = 0$ provided we define $D(1)$ to be either 0 or 1. It is also straightforward to check that both $T \circ T$ and $T \circ D$ are continuous on $[0,1]$. So what we have shown is that

$$T \circ D(x) = T \circ T(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/4 \\ 2-4x & \text{if } 1/4 \leq x \leq 1/2 \\ 4x-2 & \text{if } 1/2 \leq x \leq 3/4 \\ 4-4x & \text{if } 3/4 \leq x \leq 1 \end{cases},$$