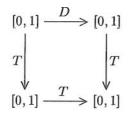
In either case, $D(x) = 0.b_2b_3b_4...$ and so we see that doubling is equivalent to the shift. Consequently, anything true (in the dynamical sense) of the shift is also true of doubling, and in particular, D is chaotic because σ is.

21. Prove that the function

$$T(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2 - 2x & \text{if } x > 1/2 \end{cases}$$

is chaotic on [0, 1].

We will show that the doubling map is semi-conjugate to the tent map T via T itself! That is, we will show that



commutes. (Note that the doubling map D needs to be defined on the closed unit interval for this to work.) Thus T will be chaotic by virtue of Exercise 20. This is because orbits under iteration of D map to dynamically equivalent orbits under T. In fact, we now prove by induction that

$$T \circ D^{n-1} = T^n \tag{10.1}$$

for all n > 0. Suppose Equation 10.1 is true for n := k. Then

$$T \circ D^{k-1} = T^k$$

$$\Rightarrow T \circ T \circ D^{k-1} = T \circ T^k,$$

and since $T \circ D = T \circ T$ (this will be verified in a moment) we have

$$T \circ D^k = T^{k+1}$$

which completes the inductive proof. We remark that (10.1) gives an explicit formula for $T^n(x)$ since we already know that $D^{n-1}(x) = 2^{n-1}x \mod 1$.

We now show that D is semi-conjugate to T via T, or in other words, that $T \circ D = T \circ T$. There are four cases to consider for $T \circ T$:

$$0 \le x \le 1/4 \qquad \Rightarrow \qquad 0 \le T(x) \le 1/2 \qquad \Rightarrow \qquad T \circ T(x) = T(2x) \\ = 2(2x) \\ = 4x \\ 1/4 \le x \le 1/2 \qquad \Rightarrow \qquad 1/2 \le T(x) \le 1 \qquad \Rightarrow \qquad T \circ T(x) = T(2x) \\ = 2 - 2(2x) \\ = 2 - 4x \\ 1/2 \le x \le 3/4 \qquad \Rightarrow \qquad 1/2 \le T(x) \le 1 \qquad \Rightarrow \qquad T \circ T(x) = T(2 - 2x) \\ = 2 - 2(2 - 2x) \\ = 4x - 2 \\ 3/4 \le x \le 1 \qquad \Rightarrow \qquad 0 \le T(x) \le 1/2 \qquad \Rightarrow \qquad T \circ T(x) = T(2 - 2x) \\ = 2(2 - 2x) \\ = 4 - 4x$$

Similarly, there are four cases for $T \circ D$:

We have to be a little bit careful at x=1/2 since D is not continuous there, and also at x=1 since we haven't yet defined D(1). But the reader may check that $T \circ D(1/2) = T \circ T(1/2) = 0$, and that $T \circ D(1) = T \circ T(1) = 0$ provided we define D(1) to be either 0 or 1. It is also straightforward to check that both $T \circ T$ and $T \circ D$ are continuous on [0,1]. So what we have shown is that

$$T \circ D(x) = T \circ T(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4\\ 2 - 4x & \text{if } 1/4 \le x \le 1/2\\ 4x - 2 & \text{if } 1/2 \le x \le 3/4\\ 4 - 4x & \text{if } 3/4 \le x \le 1 \end{cases},$$