MATH 374 Dynamical Systems

Exam #2 SOLUTIONS April 10, 2014 Prof. G. Roberts

1. Some conceptual questions: (22 pts.)

   (a) Give a sequence $s \in \Sigma_2$ that is periodic of period 5 under the shift map σ. Is s on an "attracting" cycle or "repelling" cycle? Explain.

   Answer: Any sequence of the form $\mathbf{s} = (\overline{s_0 s_1 s_2 s_3 s_4})$ will be period 5 under σ, provided that at least two of the s_i are distinct (otherwise $\mathbf{s}$ is a fixed point). For example, $\mathbf{s} = (01100\ \ \overline{01100})$ is a period 5-cycle.

   Any periodic cycle for σ is "repelling" because we can always find an arbitrarily close sequence $\mathbf{t}$ that will move away under iteration. For example, $\mathbf{t} = (01100\ \ 01100\ \ 100\cdots)$ will start within $1/2^9$ of $\mathbf{s}$, but after the tenth iterate, it will be at least $1 + 1/2 + 1/4 = 7/4$ away from $\mathbf{s}$. Thus, there does not exist a neighborhood of $\mathbf{s}$ that asymptotically approaches $\mathbf{s}$ under iteration. Alternatively, the shift map σ is chaotic, so it cannot contain any attracting cycles, otherwise we would not have any of the properties of chaos.

   (b) Arrange the natural numbers $4, 7, 24, 30$ in order from least to greatest using Sarkovskii's ordering of the natural numbers.

   Answer: Factor each number into powers of 2 times an odd number. $4 = 2^2, 24 = 2^3 \cdot 3, 30 = 2 \cdot 15$. We then have

   $$7 \ <_s\ 30 \ <_s\ 24 \ <_s\ 4$$

   using Sarkovskii's ordering of $\mathbb{N}$.

   (c) Give a precise definition of the following concept: A dynamical system $f : X \to X$ exhibits *sensitive dependence on initial conditions* if ...

   Answer: there exists a $\delta > 0$ such that for any $x \in X$ and for any $\epsilon > 0$, we can find a $y \in X$ and a $k \in \mathbb{N}$ satisfying $d(x, y) < \epsilon$ and $d(f^k(x), f^k(y)) > \delta$. In words, given any point x in X, there are points arbitrarily close to x that will move away at least a distance of δ under iteration.

   (d) Give two examples of functions that are chaotic.

   Answer: The doubling map, tripling map, and tent map are each examples of chaotic dynamical systems (provable using the graphs of their higher iterates). Also, the shift map σ on Σ_2 and $Q_c(x) = x^2 + c$ for $c < -2$ on the Cantor set Λ are each chaotic, as proved in class.

2. Recall that $Q_c(x) = x^2 + c$ and p_+ is the largest fixed point of Q_c. Suppose that $c < -2$. Let $I = [-p_+, p_+]$ and $\Gamma = \{x \in I : Q_c^n(x) \in I \ \forall n \in \mathbb{N}\}$.

Circle **all** of the following choices that are true. This is **NOT** multiple choice. There may be more than one correct answer. No work is required for this problem. (18 pts.)

(a) $0 \in \Gamma$.

(b) $-p_+ \in \Gamma$.

(c) Γ is totally disconnected.

(d) Γ is a countable set.

(e) The set Γ is dense in I.

(f) The set of periodic points of Q_c is dense in Γ.

Answer: **(b)**, **(c)** and **(f)**

Choice **(a)** is false because 0 immediately leaves I on the first iteration (the vertex of the parabola drops below the box). Choice **(b)** is true because $Q_c(-p_+) = p_+$, which is a fixed point that never escapes. Choice **(c)** is true, as proven in class, since Γ does not contain any intervals. Choice **(d)** is false because Γ is homeomorphic via the itinerary map S to the set Σ_2, which is uncountable. Choice **(e)** is false because Γ is constructed from I by repeatedly deleting the open "middle-thirds" intervals. What remains is a Cantor "dust," which is not dense. Choice **(f)** is true because Q_c on Γ is topologically conjugate to the shift map σ on Σ_2. Since σ possesses a dense set of periodic points in Σ_2, Q_c possesses a dense set of periodic points in Γ (this property is preserved under topological conjugacy, as proven in class.)

3. For the two sequences $\mathbf{s}, \mathbf{t} \in \Sigma_2$ defined as

$$\mathbf{s} = (010\,010\,\overline{010}), \qquad \mathbf{t} = (100\,100\,\overline{100}),$$

compute $d[\mathbf{s}, \mathbf{t}]$. (10 pts.)

Answer: $d[\mathbf{s}, \mathbf{t}] = 12/7$.

Using the definition of the metric d, we have that

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] &= \frac{1}{2^0} + \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \cdots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{8} + \frac{1}{16}\right) + \left(\frac{1}{64} + \frac{1}{128}\right) + \cdots \\ &= \frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \cdots \\ &= \frac{3/2}{1 - 1/8} = \frac{3/2}{7/8} = \frac{3}{2} \cdot \frac{8}{7} = \frac{12}{7}, \end{aligned}$$

using the formula for the sum of a geometric series $S = a/(1 - r)$.

4. Consider the subset T of Σ_2 defined as

$$T = \{(s_0 s_1 s_2 \ldots) : \text{ the sequence contains infinitely many 0's and infinitely many 1's } \}.$$

Decide whether or not T is dense in Σ_2. Prove your assertion. (12 pts.)

Answer: T is dense in Σ_2.

Proof: Let $\mathbf{s} = (s_0 s_1 s_2 \cdots s_n \cdots)$ be an arbitrary sequence in Σ_2. Let $\epsilon > 0$ be an arbitrary, but fixed, small distance. We must find an element $\mathbf{t} \in T$ such that $d(\mathbf{s}, \mathbf{t}) < \epsilon$. Choose $n \in \mathbb{N}$ sufficiently large so that $1/2^n < \epsilon$. Define $\mathbf{t}$ to be the sequence

$$\mathbf{t} = (s_0 s_1 s_2 \cdots s_n\, 0101\overline{01}).$$

Note that $\mathbf{t}$ contains an infinite number of 0's and an infinite number of 1's and is therefore an element of T. By the Proximity Theorem, since $t_i = s_i \ \forall i \leq n$, we have $d(\mathbf{s}, \mathbf{t}) \leq 1/2^n < \epsilon$, as desired.

5. Prove that $G : \Sigma_2 \to \Sigma_2$ given by

$$G(s_0 s_1 s_2 \ldots) = (s_0 s_1 s_4 s_9 s_{16} \ldots)$$

is a continuous function. (12 pts.)

Answer: Note that the indices for $G(\mathbf{s})$ are elements of the sequence $\{n^2\}$.

Proof: Let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ sufficiently large such that $1/2^n < \epsilon$. We must find a $\delta > 0$ such that $d(\mathbf{s}, \mathbf{t}) < \delta$ implies that $d(G(\mathbf{s}), G(\mathbf{t})) < \epsilon$.

Set $\delta = 1/2^{n^2}$. If $d(\mathbf{s}, \mathbf{t}) < \delta = 1/2^{n^2}$, then $s_i = t_i \ \forall i \leq n^2$ by the second half of the Proximity Theorem. Then, we have

$$\begin{aligned} G(\mathbf{s}) &= (s_0 s_1 s_4 s_9 s_{16} \cdots s_{n^2} \cdots) \\ G(\mathbf{t}) &= (s_0 s_1 s_4 s_9 s_{16} \cdots s_{n^2} t_{(n+1)^2} \cdots) \end{aligned}$$

so that $G(\mathbf{s})$ and $G(\mathbf{t})$ agree on their first $n+1$ entries. Using the Proximity Theorem, this implies that $d(G(\mathbf{s}), G(\mathbf{t})) \leq 1/2^n < \epsilon$. Thus, we have shown that

$$d(\mathbf{s}, \mathbf{t}) < \delta \quad \Longrightarrow \quad d(G(\mathbf{s}), G(\mathbf{t})) < \epsilon,$$

which proves that G is continuous.

6. Compute the Schwarzian derivative $SF(x)$ of $F(x) = e^{kx}$ where $k \neq 0$ is an arbitrary constant. Conclude that $SF(x) < 0 \; \forall x$. (10 pts.)

   Answer: $-\frac{1}{2}k^2$

   Using the formula

$$SF(x) \;=\; \frac{f'''(x)}{f'(x)} \;-\; \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

   we have

$$SF(x) \;=\; \frac{k^3 e^{kx}}{k e^{kx}} - \frac{3}{2}\left(\frac{k^2 e^{kx}}{k e^{kx}}\right)^2 \;=\; k^2 - \frac{3}{2}k^2 \;=\; -\frac{1}{2}k^2.$$

   Since $k \neq 0$ is assumed, we have that $k^2 > 0$, which implies that $-\frac{1}{2}k^2 < 0$ and thus $SF(x) < 0 \; \forall x$.

7. **True or False:** If true, provide a proof. If false, give an explanation as to why the statement is false. (16 pts.)

   (a) The number 1/4 is in the middle-thirds Cantor set.

   Answer: True

   The key is to write 1/4 in its ternary expansion: $1/4 = 0.020202\cdots$. This can be checked easily enough, since

$$0.\overline{02} \;=\; \frac{2}{9} + \frac{2}{81} + \frac{2}{3^6} + \cdots \;=\; \frac{2/9}{1 - 1/9} \;=\; \frac{2/9}{8/9} \;=\; \frac{2}{9} \cdot \frac{9}{8} \;=\; \frac{1}{4}.$$

   Since a point with only 0 or 2 in its ternary expansion is in the Cantor set, 1/4 is in the middle-thirds Cantor set.

   Note: To find the ternary expansion of 1/4 we begin by noting that $1/4 \in [0, 1/3]$ and $1/4 \in [2/9, 3/9]$. This means that the first two entries are 0 and then 2. Then, notice that if we triple 1/4 (mod 1), we see that $1/4 \mapsto 3/4 \mapsto 9/4 = 1/4$ so that 1/4 is on a period 2-cycle under the tripling map. This implies that its ternary expansion is also period 2, as confirmed above.

   (b) Suppose that f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and that f is topologically conjugate to g. If f has a periodic cycle of prime period 3, then g has periodic cycles of all periods.

   Answer: True

   Since f has a cycle of prime period 3, so does g, because the conjugacy between f and g maps period n points to period n points (proven on homework). Since g is continuous and has a period 3-cycle, by the Period 3 Theorem (or by Sarkovskii's Theorem), g has periodic cycles of all other periods.