MATH 374 Dynamical Systems

Exam #1 SOLUTIONS **February 27, 2014** **Prof. G. Roberts**

1. **The Attracting Fixed Point Theorem** (12 pts.)

   (a) Precisely state the Attracting Fixed Point Theorem.

   Answer: Suppose that f is a C^1 function (differentiable with a continuous derivative) and that x_0 is an attracting fixed point for f, that is, $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then there exists an open interval I containing x_0 such that for any $x \in I$, $f^n(x) \in I \; \forall n \in \mathbb{N}$, and

$$\lim_{n\to\infty} f^n(x) \;=\; x_0.$$

   (b) What important theorem from calculus is used to prove the Attracting Fixed Point Theorem? (You do **not** need to prove the theorem.)

   Answer: Mean Value Theorem

2. **Fill-in the Blanks/Multiple Choice** (20 pts.)

   (a) The quadratic family $Q_c(x) = x^2 + c$ undergoes a period-doubling bifurcation at $c =$ $\underline{-3/4 \text{ or } -5/4}$.

   (b) The name of the scientist who first noticed the Butterfly Effect while modeling the weather is Edward Lorenz.

   (c) Suppose that $F(x)$ is a differentiable function with $F(p) = p, F'(p) = 1, F''(p) = 0$ and $F'''(p) < 0$. Which one of the following is correct?

   (i) p is a weakly attracting fixed point.

   (ii) p is a weakly repelling fixed point.

   (iii) To the left, p repels nearby points under iteration, but to the right, p attracts nearby points under iteration.

   (iv) To the left, p attracts nearby points under iteration, but to the right, p repels nearby points under iteration.

   (v) Not enough information to determine the fate of orbits near p.

   Answer: (i), p is a weakly attracting fixed point. Since $F'''(p) < 0$, we know that F'' is a decreasing function at p. Since $F''(p) = 0$, we know that $F''(x) > 0$ for $x < p$ and $F''(x) < 0$ for $x > p$. Thus, F is concave up to the left of p, concave down to the right of p and has an inflection point at p where F is tangent to the diagonal $y = x$. Thus the graph of F will cross the diagonal from above, and using graphical analysis, we see that p is a weakly attracting fixed point. It is "weakly" attracting because $F'(p) = 1$.

   (d) Suppose that $f(x)$ and $g(x)$ are differentiable functions with $f(p) = p$, $f'(p) = 0.3$, $g(p) = p$ and $g'(p) = -0.3$. Which one of the following is correct?

   (i) p is an attracting fixed point for both functions, and nearby orbits converge to p faster for f than they do for g.

   (ii) p is an attracting fixed point for both functions, and nearby orbits converge to p faster for g than they do for f.

   (iii) p is an attracting fixed point for both functions, and nearby orbits converge to p at the same rate for each function, but orbits will oscillate about the fixed point for g.

(iv) p is an attracting fixed point for f and a repelling fixed point for g.

(v) p is a repelling fixed point for both functions.

Answer: (iii). This question was a review of material from the first computer lab. It is the *absolute value* of the derivative at the fixed point that controls the rate of convergence to the fixed point. The negative sign causes the orbit to oscillate about the fixed point as it approaches, as can be seen from a web diagram.

3. Use an accurate web diagram, graphical analysis, and calculus to perform a complete orbit analysis of the function

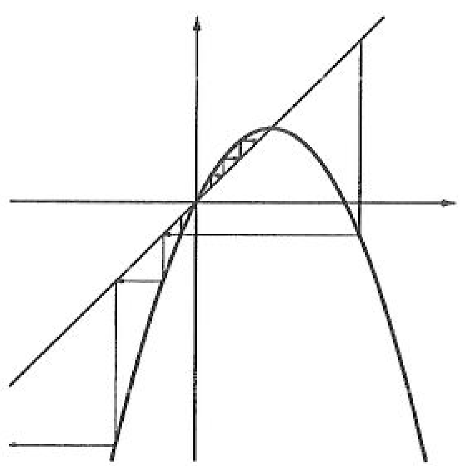
$$F(x) = 2x(1-x).$$

Be sure to describe the fate of **all** orbits, being as precise as possible. List any attracting and repelling fixed points or cycles, asymptotic orbits, orbits heading towards $+\infty$ or $-\infty$, etc. (18 pts.)

Answer:

The graph of F is a parabola opening down with roots at 0 and 1, and vertex at the point $(1/2, 1/2)$ (see figure and web diagram below). Solving $F(x) = x$ for the fixed points gives $2x(1-x) = x$. This implies that $x = 0$ or $2(1-x) = 1$. The last equation yields $x = 1/2$. We also have that $F'(x) = 2 - 4x$. Since $F'(0) = 2, x_0 = 0$ is a repelling fixed point. Since $F'(1/2) = 0, x_0 = 1/2$ is a super-attracting fixed point. This can also be seen from the web diagram.

If $0 < x_0 < 1$, then the orbit of x_0 rapidly approaches the super-attracting fixed point at 1/2. In the case where $1/2 < x_0 < 1$, the first iterate maps into the interval $(0, 1/2)$ and then remains in this interval as the orbit asymptotically (and monotonically) approaches 1/2. The boundary point $x_0 = 1$ is eventually fixed (after one iterate) on the repelling fixed point at 0. If $x_0 > 1$, the first iterate is less than zero and the orbit approaches negative infinity. If $x_0 < 0$, the orbit approaches negative infinity as well. Note that we have analyzed the fate of *all* orbits, so it follows that there are no periodic cycles of any period other than 1 (the fixed points).



(a) The graph of $F(x) = 2x(1-x)$.

4. The *Tripling Map* $T : [0, 1) \mapsto [0, 1)$ is given by $T(x) = 3x \mod 1$. This is equivalent to the piecewise linear function

$$T(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/3 \\ 3x - 1 & \text{if } 1/3 \le x < 2/3 \\ 3x - 2 & \text{if } 2/3 \le x < 1 \end{cases}$$

(a) Sketch the graphs of $T(x)$ and $T^2(x)$ on the interval $[0, 1)$. (8 pts.)

Answer:

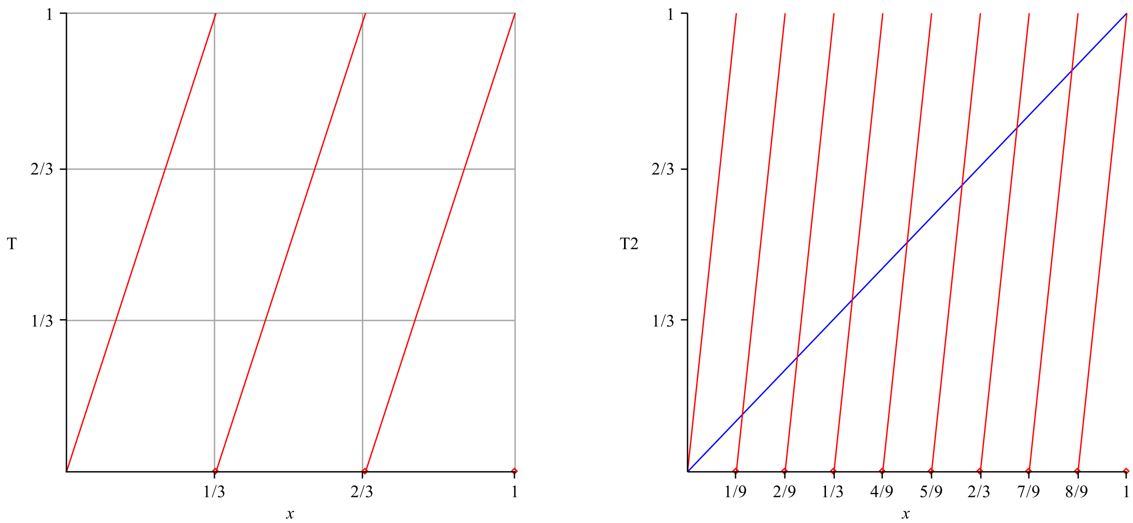


Figure 1: The graphs of $T(x)$ (left) and $T^2(x)$ (right).

(b) Find **one** prime period 2-cycle for T and classify it as attracting, repelling or neutral. (4 pts.)

Answer: To find a period 2-cycle, we must solve $T^2(x) = x$. Using the graph of T^2 above, there are eight solutions, although two of these correspond to the fixed points 0 and 1/2. For example, the second line segment in the graph of $T^2(x)$ is $9x - 1$. Solving $9x - 1 = x$ yields $x = 1/8$. Then, we check that $T(1/8) = 3/8$ and $T(3/8) = 1/8$, so that $\{1/8, 3/8\}$ is a period 2-cycle. Similarly $9x - 2 = x$ yields $x = 1/4$ and the period 2-cycle $\{1/4, 3/4\}$. The final period 2-cycle is $\{5/8, 7/8\}$, which comes from solving $9x - 5 = x$ or $9x - 7 = x$. Each of these cycles is repelling since $(T^2)'(x) = 9 > 1$ on its domain.

5. The orbit of $x_0 = 1$ under iteration of the function

$$G(x) = \begin{cases} 3x^2 + 5 & \text{if } x < 1.5 \\ \frac{1}{2}x & \text{if } x \ge 1.5 \end{cases}$$

lies on a periodic cycle. Find this cycle and determine whether it is attracting, repelling or neutral. (10 pts.)

Answer: We compute $G(1) = 8, G(8) = 4, G(4) = 2, G(2) = 1$, so that $x_0 = 1$ is on a period 4-cycle $\{1, 8, 4, 2, 1, 8, 4, 2, \ldots\}$. To check the stability type of this orbit we use the chain rule:

$$\left|(G^4)'(1)\right| = |G'(1) \cdot G'(8) \cdot G'(4) \cdot G'(2)| = \left|6 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right| = \frac{3}{4}.$$

Since $|(G^4)'(1)| < 1$, the period 4-cycle is attracting.

6. Consider the one-parameter family of dynamical systems given by

$$F_c(x) = x^2 + cx$$

where $c \in \mathbb{R}$ is a parameter. (28 pts.)

(a) Find the fixed points of F_c.

Answer: Solving $F_c(x) = x$ yields the equation $x^2 + cx = x$ or $x^2 + x(c-1) = 0$. Factoring, we obtain $x(x+c-1) = 0$, so the fixed points are $x_0 = 0$ and $x_1 = 1-c$. Note how much easier it is to factor than use the quadratic formula. Also note that there are two fixed points for all $c \in \mathbb{R}$, except for $c = 1$ where $x_0 = x_1 = 0$ is a double root of $F_1(x) - x$.

(b) Determine the values of c for which each fixed point is attracting.

Answer: We compute that $F_c'(x) = 2x + c$ so that $|F_c'(0)| = |c| < 1$ if and only if $-1 < c < 1$. Similarly, we have that $|F_c'(1-c)| = |2(1-c)+c| = |2-c| < 1$ if and only if $1 < c < 3$. Therefore, $x_0 = 0$ is attracting for $-1 < c < 1$, and $x_1 = 1-c$ is attracting for $1 < c < 3$.

(c) Find the period 2-cycle for F_{-3} (when $c = -3$).

Answer: First we compute $F_{-3}^2(x)$, then we must solve $F_{-3}^2(x) = x$, eliminating the two fixed point solutions $x_0 = 0$ and $x_1 = 1-(-3) = 4$. We have

$$F_{-3}^2(x) = (x^2-3x)^2 - 3(x^2-3x) = x^4 - 6x^3 + 6x^2 + 9x.$$

This means $F_{-3}^2(x) = x$ is equivalent to $x^4 - 6x^3 + 6x^2 + 8x = 0$. Since $x_0 = 0$ and $x_1 = 4$ are each roots of this equation (as fixed points for $c = -3$), we know that $x(x-4) = x^2 - 4x$ is a factor of this quartic. By guess and check (or long division!), we see that

$$x^4 - 6x^3 + 6x^2 + 8x = (x^2 - 4x)(x^2 - 2x - 2),$$

so the period 2-cycle is found by solving the quadratic equation $x^2 - 2x - 2 = 0$. Using the quadratic formula, we obtain the solution

$$\frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}.$$

(d) Describe the bifurcation that occurs at $c = 1$. Be sure to describe the dynamical behavior for $c < 1$, $c = 1$ and $c > 1$.

Answer: Using our solution to parts **(a)** and **(b)**, we see that for $c < 1$ (but close to 1), the fixed point at $x_0 = 0$ is attracting, while the fixed point at $x_1 = 1-c$ is repelling. At the bifurcation $c = 1$, the two fixed points *merge* into a neutral fixed point at 0 with slope $F_1'(0) = 1$. This neutral fixed point weakly attracts from the left, and weakly repels to the right (via web diagram). After the bifurcation, for $c > 1$, the fixed points have *interchanged* their stability types, with $x_0 = 0$ now becoming repelling and $x_1 = 1-c$ now becoming attracting. In summary, the fixed points x_0 and x_1 merge at the bifurcation and interchange their stability types.

Note: This is neither a saddle-node bifurcation nor a period-doubling bifurcation. It is not a saddle-node bifurcation because no new fixed points are born after the bifurcation. It also fails to satisfy the condition (at $c = 1$)

$$\frac{\partial}{\partial c}\left(F_c(x)\right)(x_0 = 0) \neq 0.$$

It is not a period-doubling bifurcation because the slope at the neutral fixed point is 1 rather than -1. A bifurcation diagram of this family is shown below. Note the bifurcation at $c = 1$ as well as the period-doubling bifurcations at $c = -1$ and $c = 3$, respectively, where $x_0 = 0$ ($x_1 = -2$, respectively) have slopes of -1.

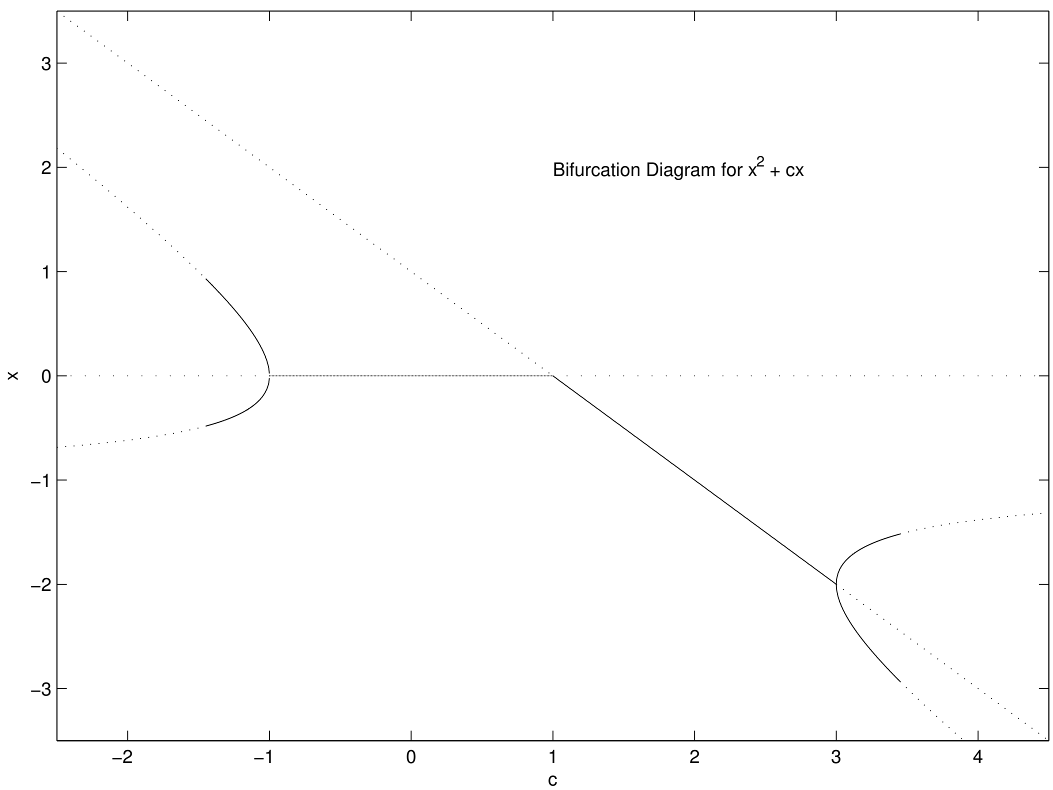


Figure 2: The bifurcation diagram for the family $F_c(x) = x^2 + cx$. Solid lines or curves indicate attracting cycles; dashed lines indicate repellers.