Every member of this family is anchored to the origin, with another zero at $x = -1/c$. Figure 6.5 shows that $\text{fix}\, H_c = \{0\}$, and that $H_c'(0) = 1$ regardless of c. Now, let's apply the results of Theorem 5.2. Since $H_c''(0) = 2c$, case 1 of the theorem applies provided $c \neq 0$. For $c < 0$, we see that $H_c''(0) < 0$, and so 0 is weakly attracting on the right and weakly repelling on the left. Similarly, for $c > 0$, $H_c''(0) > 0$, and the origin is weakly repelling on the right and weakly attracting on the left. But when $c = 0$, $H_c''(0) = 0$, and we look to case 2 of Theorem 5.2. Unfortunately, $H_c'''(x)$ is identically zero and so the theorem does not apply. Observe, however, that H_0 is the identity map, which is totally periodic.

In summary, this family of maps experiences no bifurcations whatsoever, and provides a good example of why the precise definitions given in Chapter 6 of the text are necessary.

1k) $F_c(x) = x + cx^2 + x^3, \quad c = 0$

First of all, when $c = 0$, F_0 is identical to the map in Exercise 1d with $\lambda = 1$. (See F_1 and F_1^2 in Figure 6.2a.) But a generic member of this family of functions has two fixed points (see Figures 6.6b–c) since

$$\begin{aligned} & x + cx^2 + x^3 = x \\ \Rightarrow\ & cx^2 + x^3 = 0 \\ \Rightarrow\ & (c + x)x^2 = 0 \\ \Rightarrow\ & x = -c \quad \text{or} \quad x = 0. \end{aligned}$$

Now, $F_c'(x) = 1 + 2cx + 3x^2$, and so $F_c'(0) = 1$ regardless of c. Hence, this fixed point fails to undergo a bifurcation which can be seen in Figure 6.6d.

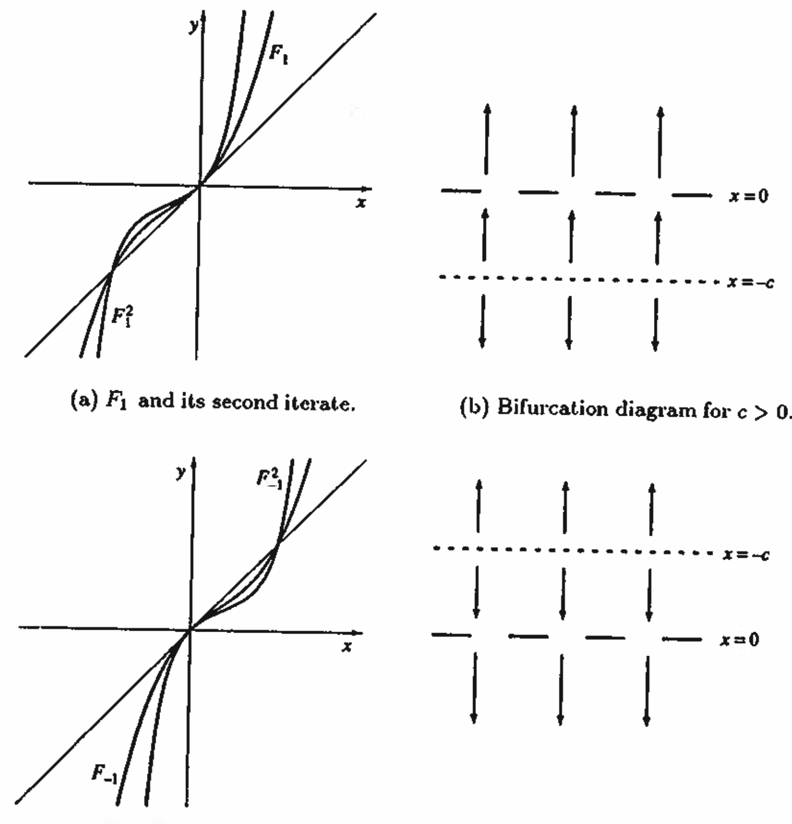
Note that $F_c'(-c) = 1 + c^2$ which is strictly greater than one for all $c \neq 0$. Thus, $-c$ is repelling for all c (even $c = 0$ which is weakly repelling). We also remark that $F_{-c}(-x) = -F_c(x)$, a most curious property.

The next four exercises apply to the family $Q_c(x) = x^2 + c$.

2. Verify the formulas for the fixed points $p_\pm$ and the 2-cycle $q_\pm$ given in the text.

Recall that

$$p_\pm = \frac{1 \pm \sqrt{1 - 4c}}{2}$$



(a) F_1 and its second iterate. (b) Bifurcation diagram for $c > 0$.

(c) F_{-1} and its second iterate. (d) Bifurcation diagram for $c < 0$.

Figure 6.6: Representatives of the family $F_c(x) = x + cx^2 + x^3$.