Exercise 5.4d that fix $S_1 = \{0\}$ and that the origin is weakly attracting; see also Exercise 4.1g. Indeed, the fact that $S_1'(0) = 1$ implies that $x = 0$ undergoes a saddle-node bifurcation at $\mu = 1$. Moreover, since $S_\mu'(0) = \mu$, the origin is attracting for $-1 < \mu < 1$ and repelling for $|\mu| > 1$.

1f) $S_\mu(x) = \mu \sin x, \quad \mu = -1$ (see Figure 6.3b)

Since $S_{-1}'(0) = -1$, it appears that S_μ undergoes a period-doubling bifurcation at $\mu = -1$. We remark that for $\mu < -1$, S_μ has an attracting 2-cycle.

The reader may wonder if there other bifurcation points for S_μ, and if so, what are they? We begin to answer this question below.

Since bifurcations occur at neutral fixed points, what we need to do is solve the equations

$$\mu \sin x = x \tag{6.1}$$

and

$$\mu \cos x = \pm 1 \tag{6.2}$$

simultaneously. Now, if we divide (6.1) by (6.2), we get

$$\tan x = \pm x$$

for $\mu \neq 0$. (The tangent function was cursorily examined earlier in Exercise 5.4e.) In other words, the bifurcation points of S_μ are the fixed points and 2-cycles of $x \mapsto \tan x$.[1]

1h) $E_\lambda(x) = \lambda(e^x - 1), \quad \lambda = -1$

Note that $E_\lambda(0) = 0$ and so $0 \in \text{fix}\, E_\lambda$. Also note that $E_\lambda'(x) = \lambda e^x = E_\lambda(x) + \lambda$. Thus, $E_\lambda'(0) = \lambda$. Therefore, the origin is attracting for $|\lambda| < 1$ and repelling for $|\lambda| > 1$. When $\lambda = -1$, E_λ undergoes a period-doubling bifurcation since $E_{-1}(0) = -1$. See Figure 6.4.

1i) $E_\lambda(x) = \lambda(e^x - 1), \quad \lambda = 1$

We have from the previous problem that $E_\lambda'(x) = E_\lambda(x) + \lambda$. It follows that $E_1'(0) = 1$ which shows there's a saddle-node bifurcation at $\lambda = 1$. See Figure 6.4 for the graph of E_1.

1j) $H_c(x) = x + cx^2, \quad c = 0$

[1] Recall that tangent is an odd function, and that the 2-cycles of an odd function F are solutions to the equation $F(x) = -x$.

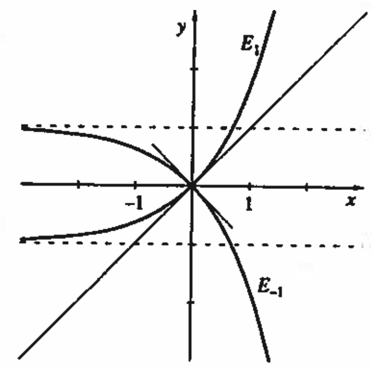

Figure 6.4: Two examples from the exponential family $E_\lambda(x) = \lambda(e^x - 1)$.

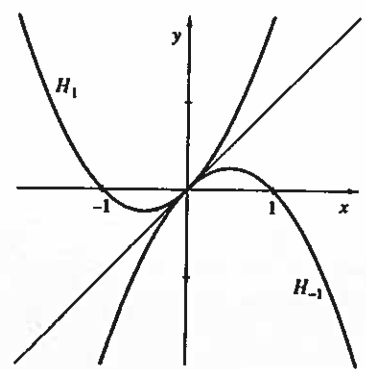

Figure 6.5: Every member of the family $H_c(x) = x + cx^2$ has a neutral fixed point.