2/3 is also repelling. In fact, no periodic point for T can be attracting! See Exercise 3 for a related result.

1k) $F(x) = 1/x^2$

fix $F = \{1\}$.

$F'(x) = -2/x^3$.

$F'(1) = -2$. Therefore, $x = 1$ is repelling.

2. For each of the following functions, zero lies on a periodic orbit. Classify this orbit as attracting, repelling, or neutral.

2a) $F(x) = 1 - x^2$

Since $F(0) = 1$ and $F(1) = 0$, $\{0, 1\} \subseteq \text{per}_2 F$.

$F'(x) = -2x$.

$(F^2)'(0) = F'(0) \cdot F'(1) = 0 \cdot (-2) = 0$, and so this period 2 orbit is superattracting.

2b) $C(x) = \frac{\pi}{2} \cos x$

Since $C(0) = \pi/2$ and $C(\pi/2) = 0$, $\{0, \pi/2\} \subseteq \text{per}_2 C$.

$C'(x) = -(\pi/2) \sin x$.

$(C^2)'(0) = C'(0) \cdot C'(\pi/2) = 0 \cdot (-\pi/2) = 0$, and once again the orbit is superattracting.

2c) $F(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$

Since $F(0) = 1$, $F(1) = -1$, and $F(-1) = 0$, we have that $\{0, \pm 1\} \subseteq \text{per}_3 F$.

$F'(x) = -\frac{3}{2}x^2 - 3x$.

$(F^3)'(0) = F'(0) \cdot F'(1) \cdot F'(-1) = 0 \cdot (-\frac{9}{2}) \cdot (\frac{3}{2}) = 0$, and so this period 3 orbit is superattracting.

2d) $F(x) = |x - 2| - 1$

Note that $F(0) = 1$ and $F(1) = 0$. Thus $0 \in \text{per}_2 F$. In fact, every point is eventually periodic with period 2.

Since

$$|x-2| - 1 = \begin{cases} x - 3 & \text{if } x \geq 2 \\ 1 - x & \text{if } x < 2 \end{cases},$$

it follows that

$$F'(x) = \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

and therefore, $(F^2)'(0) = F'(0) \cdot F'(1) = (-1) \cdot (-1) = 1$. This implies that the orbit of 0 is neutral.

2e) $A(x) = -\frac{4}{\pi} \arctan(x + 1)$

Since $A(0) = -1$ and $A(-1) = 0$, we see that $\{-1, 0\} \subseteq \text{per}_2 A$.

The reader may verify that

$$A'(x) = \frac{-4}{\pi(1 + (x+1)^2)},$$

and $(A^2)'(0) = A'(0) \cdot A'(-1) = (-2/\pi) \cdot (-4/\pi) = 8/\pi^2 < 1$. This implies that 0 is an attracting periodic point of period 2.

2f) $F(x) = \begin{cases} x + 1 & \text{if } x \leq 3.5 \\ 2x - 8 & \text{if } x > 3.5 \end{cases}$

In this case, $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 0$, and so $0 \in \text{per}_5 F$. Also,

$$F'(x) = \begin{cases} 1 & \text{if } x < 3.5 \\ 2 & \text{if } x > 3.5 \end{cases}$$

from which it follows that $(F^5)'(0) = F'(0) \cdot F'(1) \cdot F'(2) \cdot F'(3) \cdot F'(4) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$. Thus, 0 is a repelling periodic point of period 5.

3. Suppose x_0 lies on a cycle of prime period n for the doubling function D. Evaluate $(D^n)'(x_0)$. Is this cycle attracting or repelling?

Recall the definition of the doubling map given at the end of Chapter 3:

$$\begin{aligned} D(x) &= 2x \bmod 1 \\ &= \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}. \end{aligned}$$

(See Figure 5.1.) The crucial fact employed here is that for all $x \neq 1/2$, $D'(x) = 2$. Now, let $x_0 \in \text{per}_n D$, that is, suppose $D^n(x_0) = x_0$,[3] and let

[3] This periodic point can not be equal to 1/2 since 1/2 is eventually fixed. Moreover, for all $k > 0$, it must be true that $D^k(x_0) \neq 1/2$ since each such point is eventually fixed. Indeed, the reader is encouraged to write down an expression for $\overline{\text{fix}\, D}$.