and moreover, every real number is eventually periodic with period 2. To see this, first suppose that $x > 2$. Then there exists an m such that $0 < F^m(x) \leq 2$ (see the detailed argument in Exercise 3.10), and therefore, $F^m(x) \in \mathrm{per}_2 F$. Consequently, $x \in \mathrm{per}_2^m F$. In the event that $x < 0$, we have that $F(x) > 2$, and so $F(x)$ is eventually periodic via a similar argument. See Figure 4.13 for some typical orbits.

6. Consider $F(x) = x^2 - 1.1$. First find the fixed points of F. Then use the fact that these points are also solutions of $F^2(x) = x$ to find the cycle of prime period 2 for F.

We begin by computing

$$\begin{aligned} & x^2 - 1.1 = x \\ \Rightarrow\ & x^2 - x - 1.1 = 0 \\ \Rightarrow\ & x = \frac{1 \pm \sqrt{1-(4)(1)(-1.1)}}{2} \end{aligned}$$

which implies the fixed points of F are

$$\frac{1 \pm \sqrt{5.4}}{2}.$$

Just to be sure, we had better check our work:

$$\begin{aligned} F\left(\frac{1 \pm \sqrt{5.4}}{2}\right) &= \left(\frac{1 \pm \sqrt{5.4}}{2}\right)^2 - 1.1 \\ &= \frac{1 \pm 2\sqrt{5.4} + 5.4}{4} - 1.1 \\ &= \frac{6.4 \pm 2\sqrt{5.4} - 4.4}{4} \\ &= \frac{2 \pm 2\sqrt{5.4}}{4} \\ &= \frac{1 \pm \sqrt{5.4}}{2} \qquad \checkmark \end{aligned}$$

Next, let's compute the second iterate of F,

$$\begin{aligned} F^2(x) &= (x^2 - 1.1)^2 - 1.1 \\ &= x^4 - 2.2x^2 + 1.21 - 1.1 \\ &= x^4 - 2.2x^2 + 0.11, \end{aligned}$$

and *its* fixed points (which are also the period 2 points of F):

$$\begin{aligned} & x^4 - 2.2x^2 + 0.11 = x \\ \Rightarrow\ & x^4 - 2.2x^2 - x + 0.11 = 0 \\ \Rightarrow\ & (x^2 - x - 1.1)(x^2 + x - 0.1) = 0 \\ \Rightarrow\ & x^2 - x - 1.1 = 0 \quad \text{or} \quad x^2 + x - 0.1 = 0 \\ \Rightarrow\ & x = \frac{1 \pm \sqrt{5.4}}{2} \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1.4}}{2}. \end{aligned}$$

But, you ask, how could *anyone* know how to factor a messy fourth degree polynomial such as that? Well, we know that any fixed point is also a period 2 point, right?[3] This means that the solutions to $x^2 - x - 1.1 = 0$ must also be solutions to $x^4 - 2.2x^2 - x + 0.11 = 0$, and suggests we compute

$$\frac{x^4 - 2.2x^2 - x + 0.11}{x^2 - x - 1.1} = x^2 + x - 0.1$$

by polynomial long division, say. (Thought you'd never use your college algebra, eh?) Study this technique carefully—it will prove invaluable in the sequel.

7. All of the following exercises deal with the dynamics of linear functions of the form $F(x) = ax + b$ where a and b are constants.

7a) Find the fixed points of $F(x) = ax + b$.

We have that

$$\begin{aligned} & ax + b = x \\ \Rightarrow\ & b = x - ax \\ \Rightarrow\ & b = x(1-a) \\ \Rightarrow\ & \frac{b}{1-a} = x \end{aligned}$$

provided $a \neq 1$. In other words, $\mathrm{fix}\, F = \{b/(1-a)\}$.

7b) For which values of a and b does F have no fixed points?

F has no fixed point when its graph is distinct from and parallel to the diagonal line $y = x$, that is, when $a = 1$ and $b \neq 0$.

[3] Actually, a fixed point is a period n point for *any* n.