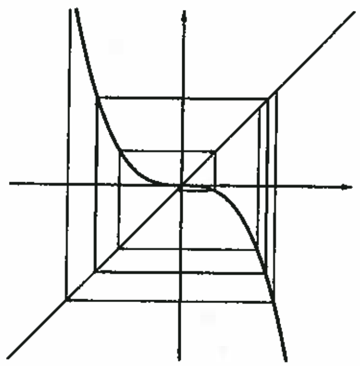

Figure 4.4: The graph of the cubic $F(x) = -x^3$.

Chapter 5), and so

$$\lim_{n\to\infty} |F^n(x)| = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1 \end{cases} . \qquad (4.2)$$

Compare (4.2) with (4.1): the absolute value signs are required in the former since points along the orbits alternate in sign.

1f) $F(x) = x - x^2$

As seen in Figure 4.3b, this function has a single fixed point at the origin. If $0 < x < 1$, then $F^n(x)$ is a positive decreasing sequence of points converging to 0. On the other hand, if $x < 0$, then $F^n(x) \to -\infty$ as $n \to \infty$. And if $x > 1$, it follows that $F(x) < 0$, and consequently, $F^n(F(x)) \to -\infty$ as $n \to \infty$. Finally, note that $x = 1$ is eventually fixed after one iteration.

Experiments indicate that for $0 < x < 1$, the orbit of x converges to 0 rather slowly. Similarly, for negative x very close to 0, $F^n(x)$ *slowly* moves away from the origin. The explanation of this behavior is that the fixed point is **neutral**, an important characteristic discussed in more detail in Chapter 5.

1g) $S(x) = \sin x$

Looking at Figure 4.5 it appears that all orbits tend to 0, which is

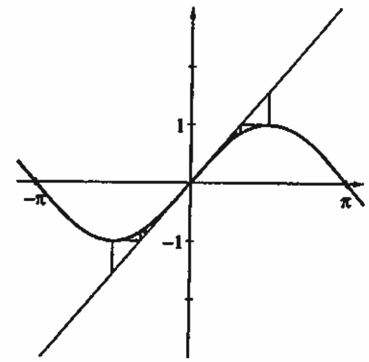

Figure 4.5: The graph of the sine function.

true, but experiments suggest that the convergence is very slow. (See Section 3.2 of the text for an illustration of this fact.)

2. Use graphical analysis to find $\{ x \mid F^n(x) \to \pm\infty \}$ for each of the following functions.

2a) $F(x) = 2x(1 - x)$ (see Figure 4.6a)

For $0 < x < 1$, $F^n(x) \to 1/2$. But when $x < 0$, $F^n(x) \to -\infty$. Now, if $x > 1$, it follows that $F(x) < 0$, and so $F^n(F(x)) \to -\infty$ as well. Thus, $\{ x \mid F^n(x) \to \pm\infty \} = \{ x \mid x < 0 \text{ or } x > 1 \}$. Note that $x = 1$ is eventually fixed after one iteration.

2b) $F(x) = x^2 + 1$ (see Figure 4.6b)

In this case, we have that $F^n(x) \to \infty$ for *all* x.

2c) $T(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2 - 2x & \text{if } x > 1/2 \end{cases}$

The map depicted in Figure 4.7 is called the **tent map**. When $x < 0$, $T^n(x) \to -\infty$. When $x > 1$, $T(x) < 0$, and so $T^n(T(x)) \to -\infty$ as $n \to \infty$ as well. But when $0 < x < 1$, $0 < T^n(x) < 1$ for *all* n. Thus, $\{ x \mid F^n(x) \to \pm\infty \} = \{ x \mid x < 0 \text{ or } x > 1 \}$. We remark that $x = 1$ is eventually fixed after one iteration since $T(1) = 0$, and 0 is fixed.

3. Sketch the phase portraits for each of the functions in Exercise 1.