

16. Sketch the graphs of T and T^2 .

See Figure 3.8 for the graph of T , and Figure 3.9a for the graph of T^2 .

17. Find all fixed points for T and T^2 .

To find the fixed points for T , simply set each piece of $T(x)$ equal to x and solve:

$$\begin{array}{ll} 2x = x & 2 - 2x = x \\ \Rightarrow 2x - x = 0 & \Rightarrow 2 = 3x \\ \Rightarrow x = 0 & \Rightarrow 2/3 = x \end{array}$$

Check:

$$\begin{array}{l} T(0) = 0 \quad \checkmark \\ T(2/3) = 2 - 2(2/3) = 2 - 4/3 = 2/3 \quad \checkmark \end{array}$$

Now set each piece of T^2 equal to x and solve:

$$\begin{array}{ll} 4x = x & 2 - 4x = x \\ \Rightarrow 3x = 0 & \Rightarrow 2 = 5x \\ \Rightarrow x = 0 & \Rightarrow 2/5 = x \end{array}$$

$$\begin{array}{ll} -2 + 4x = x & 4 - 4x = x \\ \Rightarrow 3x = 2 & \Rightarrow 4 = 5x \\ \Rightarrow x = 2/3 & \Rightarrow 4/5 = x \end{array}$$

Check:

$$\begin{array}{l} T(2/5) = 2(2/5) = 4/5 \quad \checkmark \\ T(4/5) = 2 - 2(4/5) = 2 - 8/5 = 2/5 \quad \checkmark \end{array}$$

The other two are actually fixed points for T and were checked above.

18. Find an explicit formula for $T^3(x)$ and sketch the graph of T^3 .

Half of the eight cases are worked out in detail below.

$$\begin{array}{ll} 0 \leq x \leq 1/8 & \Rightarrow 0 \leq T(x) \leq 1/4 \text{ and } 0 \leq T^2(x) \leq 1/2 \\ & \Rightarrow T^3(x) = T(T^2(x)) \\ & \quad = T(4x) \\ & \quad = 2(4x) \\ & \quad = 8x \\ 1/8 \leq x \leq 1/4 & \Rightarrow 1/4 \leq T(x) \leq 1/2 \text{ and } 1/2 \leq T^2(x) \leq 1 \\ & \Rightarrow T^3(x) = T(T^2(x)) \\ & \quad = T(4x) \\ & \quad = 2 - 2(4x) \\ & \quad = 2 - 8x \\ 1/4 \leq x \leq 3/8 & \Rightarrow 1/2 \leq T(x) \leq 3/4 \text{ and } 1/2 \leq T^2(x) \leq 1 \\ & \Rightarrow T^3(x) = T(T^2(x)) \\ & \quad = T(2 - 4x) \\ & \quad = 2 - 2(2 - 4x) \\ & \quad = -2 + 8x \\ 3/8 \leq x \leq 1/2 & \Rightarrow 3/4 \leq T(x) \leq 1 \text{ and } 0 \leq T^2(x) \leq 1/2 \\ & \Rightarrow T^3(x) = T(T^2(x)) \\ & \quad = T(2 - 4x) \\ & \quad = 2(2 - 4x) \\ & \quad = 4 - 8x \end{array}$$

The student should complete the remaining four cases. Ultimately, we get the following monstrous expression for $T^3(x)$:

$$T^3(x) = \begin{cases} 8x & \text{if } 0 \leq x < 1/8 \\ 2 - 8x & \text{if } 1/8 \leq x < 1/4 \\ 8x - 2 & \text{if } 1/4 \leq x < 3/8 \\ 4 - 8x & \text{if } 3/8 \leq x < 1/2 \\ 8x - 4 & \text{if } 1/2 \leq x < 5/8 \\ 6 - 8x & \text{if } 5/8 \leq x < 3/4 \\ 8x - 6 & \text{if } 3/4 \leq x < 7/8 \\ 8 - 8x & \text{if } 7/8 \leq x < 1 \end{cases}$$

Notice how every other piece of this multi-part function agrees exactly with the corresponding piece of $D^3(x)$ computed in Exercise 12 (see Equation 3.2). See Figure 3.9b for the graph of T^3 .

19. What does the graph of T^n look like?