But you may be surprised to learn that *every* rational number in $[0,1)$ is eventually periodic. And even more interesting is the fact that few computer programs are able to find them! See Experiment 3.6 in the text for an explanation of this strange phenomenon.

The following five exercises deal with the function

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2-2x & \text{if } 1/2 < x \le 1 \end{cases}.$$

The function T is called a **tent map** because of the shape of its graph on the interval $[0,1]$. See Figure 3.8.

15. Find a formula for $T^2(x)$.

The trick is to partition $[0,1]$ into four equal length subintervals:

$$\begin{aligned}
0 \le x < 1/4 \quad &\Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) \\
&\qquad = T(2x) \\
&\qquad = 2(2x) \\
&\qquad = 4x \\
1/4 \le x < 1/2 \quad &\Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad T^2(x) = T(T(x)) \\
&\qquad = T(2x) \\
&\qquad = 2-2(2x) \\
&\qquad = 2-4x
\end{aligned}$$

$$\begin{aligned}
1/2 \le x < 3/4 \quad &\Rightarrow \quad 1/2 \le T(x) \le 1 \quad \Rightarrow \quad T^2(x) = T(T(x)) \\
&\qquad = T(2-2x) \\
&\qquad = 2-2(2-2x) \\
&\qquad = 4x-2 \\
3/4 \le x < 1 \quad &\Rightarrow \quad 0 \le T(x) \le 1/2 \quad \Rightarrow \quad T^2(x) = T(T(x)) \\
&\qquad = T(2-2x) \\
&\qquad = 2(2-2x) \\
&\qquad = 4-4x
\end{aligned}$$

Thus, we have shown that

$$T^2(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4 \\ 2-4x & \text{if } 1/4 \le x \le 1/2 \\ 4x-2 & \text{if } 1/2 \le x \le 3/4 \\ 4-4x & \text{if } 3/4 \le x \le 1 \end{cases}.$$

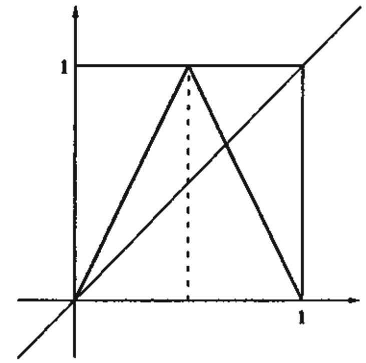
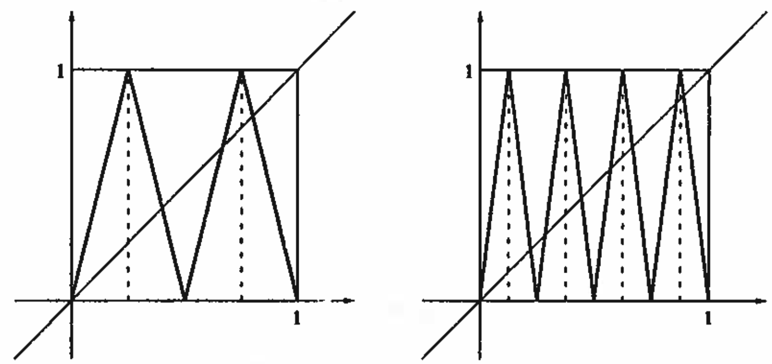


Figure 3.8: The tent map $T(x) = 1 - |2x - 1|$.



(a) Graph of T^2. (b) Graph of T^3.

Figure 3.9: The second and third iterates of the tent map.