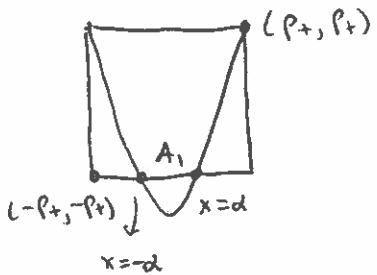


Ch. 7

(8) If $c < -\frac{(5+2\sqrt{5})}{4}$, then $|Q_c'(x)| > 1 \quad \forall x \in I - A_1$.

Claim: It suffices to show that $c < \frac{-(5+2\sqrt{5})}{4} \Rightarrow Q_c(t_2) < -p_+$.

Why?



Note that $Q_c'(x) = 2x$ so that if

$|x| > t_2$, then $|Q_c'(x)| = |2x| > 1$.

If we can show $[-t_2, t_2] \subset A_1$, then

$\forall x \in I - A_1$, $|x| > t_2$ and $|Q_c'(x)| > 1$.

But, from the graph of Q_c for $c < -2$, if

$Q_c(t_2)$ is outside the box, i.e. $Q_c(t_2) < -p_+$, then

$x = t_2 \notin A_1$ (i.e. it is removed from I on the first iteration.) Since $Q_c'' = 2 > 0$ and since

Q_c is an even function, we would then have $[-t_2, t_2] \subset A_1$.

Suppose $c < -2$. We will show directly, keeping track of inequalities,

that $Q_c(t_2) < -p_+$ iff $c < \frac{-(5+2\sqrt{5})}{4}$. Recall: $p_+ = \frac{1}{2}(1 + \sqrt{1-4c})$

$$Q_c(t_2) < -p_+ \text{ iff } \frac{1}{4}u + c < -\frac{1}{2} - \frac{1}{2}\sqrt{1-4c}$$

$$\text{iff } \frac{3}{4}u + c < -\frac{1}{2}\sqrt{1-4c}$$

$$\text{iff } -3 - 4c > 2\sqrt{1-4c} \quad (\text{mult. by } -4 \text{ reverses } <)$$

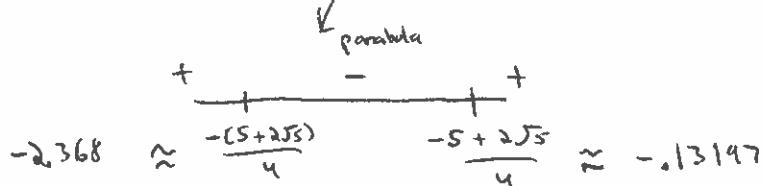
$$\text{iff } 9 + 24c + 16c^2 > 4(1-4c) \quad (\text{iff because both sides of inequality are positive})$$

$$\text{iff } 16c^2 + 40c + 5 > 0$$

$$\text{iff } c < \frac{-(5+2\sqrt{5})}{4}$$

■

↙
parabola



Since we assumed $c < -2$, we must have $c < \frac{-(5+2\sqrt{5})}{4}$.

(14) Claim: Let $x \in [0, 1]$. If x has a ternary expansion containing only 0's and 2's, then $x \in \Gamma = \{x \in [0, 1] : T(x) \in [0, 1] \text{ and } T(x) \in \Gamma\}$. Since $K = \text{Cantor middle-thirds set}$ is also equal to the set of all $x \in [0, 1]$ which have a ternary expansion containing only 0's and 2's, this shows that $\Gamma = K$.

Pf/ The proof of the claim rests on finding the ternary expansion of $T(x)$.

i) Suppose $x \leq \gamma_2$ and $x = 0.a_1 a_2 a_3 \dots$

$$= \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \dots$$

If $a_1 = 1$, then $T(x) \geq 1$ and we know $x \notin \Gamma$ (assuming $x \neq \gamma_2$)

If $a_1 = 2$, then $x \geq \gamma_3$ contradicting $x \leq \gamma_2$.

Assume $a_1 = 0$. Then $T(x) = 3x = 3 \left(\frac{0}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \dots \right)$

$$= \frac{a_2}{3} + \frac{a_3}{9} + \dots$$

\therefore The ternary expansion of $T(x)$ is $0.a_2 a_3 a_4 \dots$

ii) Suppose $x \in (\gamma_2, 1)$, and $x = 0.a_1 a_2 a_3 \dots$

$$= \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \dots$$

As before, we may assume $a_1 = 2$.

Consider the number $1-x \in (0, \gamma_2)$. Denote the ternary expansion

$$\text{of } 1-x \text{ as } 0.\hat{a}_1 \hat{a}_2 \hat{a}_3 \dots = \frac{\hat{a}_1}{3} + \frac{\hat{a}_2}{9} + \frac{\hat{a}_3}{27} + \dots$$

Thus, on the one-hand, $x + 1-x = 1 = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots = 0.\bar{222}$

$$\text{and on the other-hand, } x + 1-x = \frac{a_1 + \hat{a}_1}{3} + \frac{a_2 + \hat{a}_2}{9} + \frac{a_3 + \hat{a}_3}{27} + \dots$$

$$\therefore a_i + \hat{a}_i = 2 \quad \forall i \Rightarrow \hat{a}_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 1 & \text{if } a_i = 1 \\ 2 & \text{if } a_i = 0 \end{cases}$$

In other words, the ternary expansion of $1-x$ is the same as that of x except every 0 is replaced by a 2 and every 2 is replaced by a 0.

$$\text{eg. } \frac{1}{4} = 0.0202\overline{02} \quad \text{and} \quad 1 - \frac{1}{4} = \frac{3}{4} = 0.2020\overline{20}$$

$$\frac{1}{3} = 0.1000\overline{0} \quad \text{and} \quad 1 - \frac{1}{3} = \frac{2}{3} = 0.1222\overline{25}$$

From this, we have that $T(x) = 3 - 3x = 3(1-x)$

$$= 3 \left(\frac{0}{3} + \frac{\hat{a}_2}{9} + \frac{\hat{a}_3}{27} + \dots \right)$$

$$= \frac{\hat{a}_2}{3} + \frac{\hat{a}_3}{9} + \dots$$

\therefore The ternary expansion of $T(x)$ is $0. \hat{a}_2 \hat{a}_3 \hat{a}_4 \dots$

Putting i) and ii) together gives (assuming $a_1 \neq 1$)

$$T(x) = \begin{cases} 0. a_2 a_3 a_4 \dots & \text{if } 0 \leq x \leq y_a \\ 0. \hat{a}_2 \hat{a}_3 \hat{a}_4 \dots & \text{if } k < x \leq 1 \end{cases}$$

In other words, $T(x)$ shifts the itinerary of x to the left and either switches all 0's and 2's or fixes them.

Either way, if x contains a 1 in the i^{th} spot of its ternary expansion, i.e. if $\exists i \in \mathbb{N}$ s.t. $a_i = 1$, then

$$T^{i-1}(x) = 0.1 \hat{a}_{i+1} \hat{a}_{i+2} \dots \in (k_3, y_3)$$

and $T^i(x) \notin [0, 1]$. ■

Ch7

(16) From Hw #4, problem #4, $Q_c \sim F_\lambda$ via the conjugacy $h(x) = -\frac{1}{\lambda}x + \frac{1}{2}$, where $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$ is the relationship between the parameter values c and λ , valid for $c \leq \frac{\lambda}{4}$.

If $\lambda = 4$, then $c = -2$ and $h(x) = -\frac{1}{4}x + \frac{1}{2}$.

Thus $Q_{-2} \sim F_4$ via $h(x) = -\frac{1}{4}x + \frac{1}{2}$. Moreover, the interval $[-2, 2]$ for Q_{-2} is mapped by $h(x)$ homeomorphically onto $[0, 1]$. ($h(-2) = 1$, $h(2) = 0$)

From class, Q_{-2} has 2^n periodic points in $[-2, 2] \Rightarrow F_4$ has 2^n periodic points in $[0, 1]$.

Additional Problem $S^1: \Sigma_2 \rightarrow \Lambda$ is continuous.

Pick an arbitrary $s = (s_0 s_1 \dots s_n \dots) \in \Sigma_2$ and let $x = S^1(s) \in \Lambda$.

Let $\epsilon > 0$ be given. We know that $x \in I_{s_0 s_1 \dots s_n} \forall n$ and that these sets are shrinking in size as $n \rightarrow \infty$. (we don't know how fast they are shrinking, just that their length $\rightarrow 0$ as $n \rightarrow \infty$.)

Choose n sufficiently large such that $I_{s_0 s_1 \dots s_n} \subset (x-\epsilon, x+\epsilon)$.

Set $\delta = \frac{1}{2^n}$.

If $d(s, t) < \delta = \frac{1}{2^n}$, then $s_i = t_i \forall i \leq n$ by the Proximity Thm.

This means that $S^1(t) \in I_{s_0 s_1 \dots s_n}$ by definition of $I_{s_0 s_1 \dots s_n}$.

(The itinerary of $S^1(t)$ agrees with $x = S^1(s)$ in the first $n+1$ entries.)

$\therefore S^1(t) \in (x-\epsilon, x+\epsilon) \Rightarrow |S^1(s) - S^1(t)| < \epsilon$.

$\therefore d(s, t) < \delta \Rightarrow |S^1(s) - S^1(t)| < \epsilon$. ■