MATH 374 Dynamical Systems

Exam #2 Solutions **April 15, 2010** **Prof. G. Roberts**

1. Give precise definitions of the following: (12 pts.)

   (a) A function h is a homeomorphism if ...

   Answer: h is one-to-one, onto, continuous and has a continuous inverse.

   (b) A dynamical system $f : X \to X$ exhibits sensitive dependence on initial conditions if ...

   Answer: there exists a $\delta > 0$, such that for any $x \in X$ and any $\epsilon > 0$, we can find a $y \in X$ and a $k \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^k(x), f^k(y)) > \delta$. In words, given any point x in X, there are points arbitrarily close to x that will move away at least a distance of δ under iteration.

2. Recall that $Q_c(x) = x^2 + c$ and p_+ is the largest fixed point of Q_c. Suppose that $c < -2$. Let $I = [-p_+, p_+]$ and $\Gamma = \{x \in I : Q_c^n(x) \in I \;\; \forall n \in \mathbb{N}\}$. Circle **all** of the following choices that are true. This is **NOT** multiple choice. There may be more than one correct answer. No work is required for this problem. (18 pts.)

   (a) $0 \in \Gamma$.

   (b) Γ is totally disconnected.

   (c) Γ is a countable set.

   (d) The set Γ is dense in I.

   (e) The set of periodic points of Q_c is dense in Γ.

   (f) Q_c is chaotic on Γ.

   Answer: **(b)**, **(e)** and **(f)**

   Choice **(a)** is false because 0 immediately leaves I on the first iteration (the vertex of the parabola drops below the box). Choice **(b)** is true, as proven in class, since Γ does not contain any intervals. Choice **(c)** is false because Γ is homeomorphic via the itinerary map S to the set Σ_2 which is uncountable. Choice **(d)** is false because Γ is constructed from I by repeatedly deleting the open "middle thirds" intervals. What remains is a Cantor "dust" which is not dense. Choice **(e)** is true because Q_c on Γ is topologically conjugate to the shift map σ on Σ_2. Since σ possesses a dense set of periodic points in Σ_2, Q_c possesses a dense set of periodic points in Γ (this property is preserved under topological conjugacy, as proven in class.) Choice **(f)** is true because Q_c is topologically conjugate to the chaotic shift map σ.

3. For the two sequences $\mathbf{s}, \mathbf{t} \in \Sigma_2$ defined as

$$\mathbf{s} \;=\; (010\,010\,\overline{010}), \qquad \mathbf{t} \;=\; (100\,100\,\overline{100}),$$

compute $d[\mathbf{s}, \mathbf{t}]$. (10 pts.)

Answer: $d[\mathbf{s}, \mathbf{t}] = 12/7$.

Using the definition of the metric d, we have that

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] &= \frac{1}{2^0} + \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \cdots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{8} + \frac{1}{16}\right) + \left(\frac{1}{64} + \frac{1}{128}\right) + \cdots \\ &= \frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \cdots \\ &= \frac{3/2}{1 - 1/8} = \frac{3}{2} \cdot \frac{8}{7} = \frac{12}{7} \end{aligned}$$

using the formula for the sum of a geometric series $S = a/(1-r)$.

4. Consider the subset T of Σ_2 defined as

$$T = \{(s_0 s_1 s_2 \ldots) : \text{ the sequence contains infinitely many 0's and infinitely many 1's } \}.$$

Decide whether or not T is dense in Σ_2. Prove your assertion. (12 pts.)

Answer: T is dense in Σ_2. **Proof:** Let $\mathbf{s} = (s_0 s_1 s_2 \cdots s_n \cdots)$ be an arbitrary element of Σ_2. Let $\epsilon > 0$ be an arbitrary, but fixed, small distance. We must find an element t in T such that $d[\mathbf{s}, \mathbf{t}] < \epsilon$.

Choose $n \in \mathbb{N}$ sufficiently large so that $\frac{1}{2^n} < \epsilon$. Define $\mathbf{t}$ to be the sequence

$$\mathbf{t} = (s_0 s_1 s_2 \cdots s_n 010101\overline{01}).$$

Note that $\mathbf{t}$ contains an infinite number of 0's and an infinite number of 1's and is therefore an element of T. By the Proximity Theorem, since $\mathbf{s}$ and $\mathbf{t}$ agree on the first $n+1$ entries, $d[\mathbf{s}, \mathbf{t}] \le \frac{1}{2^n} < \epsilon$, as desired.

5. Prove that $G : \Sigma_2 \to \Sigma_2$ given by

$$G(s_0 s_1 s_2 \ldots) \;=\; (s_1 s_3 s_9 s_{27} s_{81} \ldots)$$

is a continuous function. (12 pts.)

Answer: Let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ sufficiently large so that $\frac{1}{2^n} < \epsilon$. Set $\delta = \dfrac{1}{2^{3^n}}$. Then, $d[\mathbf{s}, \mathbf{t}] < \delta = \dfrac{1}{2^{3^n}}$ implies that $s_i = t_i \;\forall i \le 3^n$ by the second half of the Proximity Theorem. Then, we have

$$\begin{aligned} G(\mathbf{s}) &= (s_1 s_3 s_9 s_{27} \cdots s_{3^n} \cdots) \\ G(\mathbf{t}) &= (s_1 s_3 s_9 s_{27} \cdots s_{3^n} t_{3^{n+1}} \cdots) \end{aligned}$$

so that $G(\mathbf{s})$ and $G(\mathbf{t})$ agree on their first $n+1$ entries. Using the Proximity Theorem, this means that $d[G(\mathbf{s}), G(\mathbf{t})] \le \frac{1}{2^n} < \epsilon$, as desired.

6. Compute the Schwarzian derivative $SF(x)$ of $F(x) = e^{kx}$ where $k \neq 0$ is an arbitrary constant. Conclude that $SF(x) < 0 \; \forall x$. (12 pts.)

   Answer: $-\frac{1}{2}k^2$

   Using the formula
   $$SF(x) \;=\; \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2,$$
   we have
   $$SF(x) \;=\; \frac{k^3 e^{kx}}{k e^{kx}} - \frac{3}{2}\left(\frac{k^2 e^{kx}}{k e^{kx}}\right)^2 \;=\; k^2 - \frac{3}{2}k^2 \;=\; -\frac{1}{2}k^2.$$

   Since $k \neq 0$ is assumed, we have that $k^2 > 0$ which implies that $-\frac{1}{2}k^2 < 0$ and thus $SF(x) < \; 0 \; \forall x$.

7. **True or False:** If true, provide a proof. If false, provide a counterexample or give an explanation as to why the statement is false. (24 pts.)

   (a) The point with ternary expansion $0.02002000200002\ldots$ is in the Cantor middle-thirds set.

   Answer: True. As proven in class, a point x is in the middle-thirds Cantor set if and only if it has a ternary expansion containing only 0's and 2's. Since the point in question has only 0's and 2's in its ternary expansion, it must be in the Cantor middle-thirds set.

   (b) Suppose that J is a closed interval on the real line and that $f : J \to J$ is a continuous function. If f has a periodic point of prime period 88, then f must also have a periodic point of prime period 20.

   Answer: False. We have that $20 = 2^2 \cdot 5$ while $88 = 2^3 \cdot 11$. It follows that $20 <_s 88$ in the Sarkovskii ordering. By the converse to Sarkovskii's Theorem, it is possible to construct a continuous function with a periodic point of prime period 88 without having a period 20 point.

   (c) Suppose that f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and that f is topologically conjugate to g. If f has a periodic cycle of prime period 3, then g has periodic cycles of all periods.

   Answer: True. Since f has a cycle of prime period 3, so does g because the conjugacy between f and g maps period n points to period n points (proven on homework). Since g is continuous and has a period 3 cycle, by the Period 3 Theorem (or by Sarkovskii's Theorem), g has periodic cycles of **all** other periods.