MATH 374 Dynamical Systems

Exam #1 Solutions **February 25, 2010** **Prof. G. Roberts**

1. **The Fixed Point Theorem** (12 pts.)

   (a) Precisely state the Fixed Point Theorem.

   Answer: If $f : [a, b] \to [a, b]$ is a continuous function, then f has a fixed point c in $[a, b]$, that is, $f(c) = c$.

   (b) What important theorem from calculus is used to prove the Fixed Point Theorem? (You do **not** need to prove the Fixed Point Theorem.)

   Answer: The Intermediate Value Theorem

2. **Fill-in the Blanks/Multiple Choice** (20 pts.)

   (a) The quadratic family $Q_c(x) = x^2 + c$ undergoes a saddle-node bifurcation at $c = \underline{\ 1/4\ }$.

   (b) The name of the scientist who first noticed the Butterfly Effect while modeling the weather is <u>Edward Lorenz</u>.

   (c) Suppose that $g(x)$ is a differentiable function with $g(p) = p, g'(p) = 1, g''(p) = 0$ and $g'''(p) < 0$. Which one of the following is correct?

   (i) p is a weakly attracting fixed point.

   (ii) p is a weakly repelling fixed point.

   (iii) To the left, p repels nearby points under iteration, but to the right, p attracts nearby points under iteration.

   (iv) To the left, p attracts nearby points under iteration, but to the right, p repels nearby points under iteration.

   (v) Not enough information to determine the fate of orbits near p.

   Answer: (i) p is a weakly attracting fixed point. Since $g'''(p) < 0$, we know that g'' is a decreasing function at p. Since $g''(p) = 0$, we know that $g''(x) > 0$ for $x < p$ and $g''(x) < 0$ for $x > p$. Thus, g is concave up to the left of p, concave down to the right of p and has an inflection point at p where g is tangent to the diagonal $y = x$. Thus the graph of g will cross the diagonal from above, and using graphical analysis, we see that p is a weakly attracting fixed point.

   (d) Suppose that $f(x)$ and $g(x)$ are differentiable functions with $f(p) = p$, $f'(p) = 0.3$, $g(p) = p$ and $g'(p) = -0.3$. Which one of the following is correct?

   (i) p is an attracting fixed point for both functions, and nearby orbits converge to p faster for f than they do for g.

   (ii) p is an attracting fixed point for both functions, and nearby orbits converge to p faster for g than they do for f.

   (iii) p is an attracting fixed point for both functions, and nearby orbits converge to p at the same rate for each function but orbits will oscillate about the fixed point for g.

(iv) p is an attracting fixed point for f and a repelling fixed point for g.

(v) p is a repelling fixed point for both functions.

Answer: (iii) This was a review of material from Lab #1. It is the *absolute value* of the derivative at the fixed point that controls the rate of convergence to the fixed point. The negative sign causes the orbit to oscillate about the fixed point as it approaches, as can be seen from a web diagram.

3. Use an accurate web diagram and graphical analysis to perform a complete orbit analysis of the function

$$F(x) \;=\; 2x(1-x)\,.$$

Be sure to describe the fate of **all** orbits, being as precise as possible. List attracting and repelling cycles, asymptotic orbits, orbits heading towards $+\infty$ or $-\infty$, etc. (18 pts.)

Answer: The graph of F is a parabola opening down with roots at 0 and 1 and vertex at the point $(1/2, 1/2)$. Solving $F(x) = x$ for the fixed points gives $2x(1-x) = x$. This implies that $x = 0$ or $2(1-x) = 1$. The last equation yields $x = 1/2$. We also have that $F'(x) = 2(1-2x)$. Since $F'(0) = 2$, $x_0 = 0$ is a repelling fixed point. Since $F'(1/2) = 0$, $x_0 = 1/2$ is a super-attracting fixed point. This can also be seen from the web diagram.

If $0 < x_0 < 1$, then the orbit of x_0 rapidly approaches the super-attracting fixed point at $1/2$. In the case where $1/2 < x_0 < 1$, the first iterate maps into the interval $(0, 1/2)$ and then remains in this interval as the orbit asymptotically approaches $1/2$. The boundary point $x_0 = 1$ is eventually fixed (after one iterate) on the repelling fixed point at 0. If $x_0 > 1$, the first iterate is less than zero and the orbit approaches negative infinity. If $x_0 < 0$, the orbit approaches negative infinity as well.

4. The orbit of $x_0 = 1$ under iteration of the function

$$G(x) = \begin{cases} 3x^2 + 5 & \text{if } x < 1.5 \\ \frac{1}{2}x & \text{if } x \geq 1.5 \end{cases}$$

lies on a periodic cycle. Find this cycle and determine whether it is attracting, repelling or neutral. (10 pts.)

Answer: We compute $G(1) = 8, G(8) = 4, G(4) = 2, G(2) = 1$, so that $x_0 = 1$ is on a period 4-cycle $\{1, 8, 4, 2, 1, 8, 4, 2, \ldots\}$. To check the stability type of this orbit we use the chain rule:

$$|(G^4)'(1)| \;=\; |G'(1) \cdot G'(8) \cdot G'(4) \cdot G'(2)| \;=\; \left| 6 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \right| \;=\; \frac{3}{4}.$$

Since $|(G^4)'(1)| < 1$, the periodic orbit is attracting.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function on $\mathbb{R}$ and that p is a critical point of f. Suppose also that $f^n(p) = p$ where $n = 9$ is the smallest natural number for which this equation is true. Prove that p lies on a super-attracting cycle of period 9 for f. (10 pts.)

Answer: There are two things to show here. First, that p is a periodic point of period 9 and second, that the orbit p lies on is super-attracting.

Since $f^9(p) = p$, we know that p is on a period 9 cycle. Moreover, since $n = 9$ is the least such natural number for which this equation is true, we know that p is not a fixed point or a period-3 point. It has *prime* period 9.

Since p is a critical point and f is differentiable, we have that $f'(p) = 0$. Using the chain rule, we have that

$$|(f^9)'(p)| \;=\; |f'(p) \cdot f'(f(p)) \cdot f'(f^2(p)) \cdots f'(f^8(p))| \;=\; |0 \cdot \text{constant}\;| \;=\; 0,$$

so that p lies on a super-attracting period 9 cycle.

6. Consider the one-parameter family of dynamical systems given by

$$F_c(x) \;=\; x^2 + cx$$

where $c \in \mathbb{R}$ is a parameter. (30 pts.)

(a) Find the fixed points of F_c.

Answer: Solving $F_c(x) = x$ yields the equation $x^2 + cx = x$ or $x(x+c) = x$. This means that $x = 0$ or $x + c = 1$, that is, $x = 1 - c$. There are two fixed points at 0 and $1 - c$. Note that this is true for *any* value of $c \in \mathbb{R}$ except for $c = 1$, where there is precisely one fixed point at 0 (double root of $F_1(x) = x$).

(b) Determine the values of c for which each fixed point is attracting.

Answer: We compute that $F_c'(x) = 2x + c$ so that $|F_c'(0)| = |c| < 1$ if and only if $-1 < c < 1$. Similarly, we have that

$$|F_c'(1-c)| \;=\; |2(1-c) + c| \;=\; |2 - c| \;<\; 1 \text{ if and only if } 1 < c < 3.$$

Therefore, 0 is attracting for $-1 < c < 1$ and $1-c$ is attracting for $1 < c < 3$. (Note that even though 0 is super-attracting when $c = 0$ and $1 - c$ is super-attracting when $c = 2$, we still include these parameter values since they are locations where the respective fixed points are "attracting.")

(c) Compute $F_c^2(x)$.

Answer: We have that

$$\begin{aligned}
F_c^2(x) &= F_c(F_c(x)) \\
&= F_c(x^2 + cx) \\
&= (x^2 + cx)^2 + c(x^2 + cx) \\
&= x^4 + 2cx^3 + c^2x^2 + cx^2 + c^2x \\
&= x^4 + 2cx^3 + (c^2 + c)x^2 + c^2x
\end{aligned}$$

(d) Find the period 2-cycle for F_4 (when $c = 4$).

Answer: Using our answer to part **(c)**, we have that $F_4^2(x) = x^4 + 8x^3 + 20x^2 + 16x$. The equation $F_4^2(x) = x$ for finding period 2 cycles is then

$$x^4 + 8x^3 + 20x^2 + 15x \;=\; 0.$$

Since both fixed points 0 and $1 - 4 = -3$ satisfy this equation, we can factor out an x and an $x + 3$ term. This factorization yields

$$x(x+3)(x^2 + 5x + 5) \;=\; 0$$

which can be found using trial and error or by using polynomial division. The period two cycle must then be the roots of $x^2 + 5x + 5$ which, by the quadratic formula, are

$$\frac{-5 \pm \sqrt{5}}{2}.$$

(e) Describe the bifurcation that occurs at $c = 1$ and if possible, give the type of bifurcation. Be sure to describe the dynamical behavior for $c < 1$, $c = 1$ and $c > 1$.

Answer: Using our solution to parts **(a)** and **(b)**, we see that for $c < 1$ (but close to 1), the fixed point at 0 is attracting while the fixed point at $1 - c$ is repelling. At the bifurcation $c = 1$, the two fixed points merge into a neutral fixed point at 0 with slope $F_1'(0) = 1$. After the bifurcation, for $c > 1$, the fixed points have interchanged their stability types, with 0 now becoming repelling and $1 - c$ now becoming attracting. This is neither a saddle-node bifurcation nor a period-doubling bifurcation. It is not a saddle-node bifurcation because no new fixed points are born after the bifurcation. It also fails to satisfy the requirement

$$\frac{\partial}{\partial c} F_c|_{c=1}(x = 0) \neq 0.$$

It is not a period-doubling bifurcation because the slope at the neutral fixed point is 1 rather than -1.