MATH 373 Sample Final Exam

Principles and Techniques of Applied Mathematics

Spring 2005 Prof. G. Roberts

1. Find the general solution to $u_x + 4u_y + u = e^{x-y}$ using the coordinate method. Find the solution satisfying $u(x,0) = 0$.

2. Consider the wave equation (modeling a very long vibrating string, for example)

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty$$

with initial conditions $u(x,0) = \phi(x) = 0$, and $u_t(x,0) = \psi(x)$ where

$$\psi(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Without calculating the solution, explain why $u(0,t) = 0 \ \forall t$.

(b) Sketch the profile of the string (u versus x) at times $t = 0, 1, 2$.

3. List three ways in which solutions of the wave equation and the diffusion equation are qualitatively different from each other.

4. Solve the following diffusion equation on the half-line:

$$u_t = ku_{xx}, \quad 0 < x < \infty, \ t > 0$$

satisfying $u(x,0) = e^{-x}$ and $u_x(0,t) = 0$. Express your solution in terms of $\mathcal{E}\text{rf}(x)$.

5. Use the Maximum Principle to prove that solutions to the Diffusion equation with inhomogeneous Dirichlet boundary conditions are unique.

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } 0 < x < L, \ t > 0, \\ u(0,t) &= g(t), \\ u(L,t) &= h(t), \\ u(x,0) &= \phi(x). \end{aligned}$$

6. Solve $u_{tt} = 4u_{xx} + 1 - 4x$ on the whole real line satisfying the initial conditions $u(x,0) = x^3/6$ and $u_t(x,0) = 0$.

7. Consider the diffusion equation $u_t = ku_{xx}$ where $u(x,t)$ gives the temperature of a uniform rod, $0 < x < L$, with some prescribed boundary conditions. Suppose that separation of variables leads to the following eigenvalue problem:

$$\begin{aligned} X'' + \lambda X &= 0, \\ X'(0) - a_0 X(0) &= 0, \\ X'(L) + a_L X(L) &= 0. \end{aligned}$$

(a) Show that if $a_0 < 0$ and $a_L < 0$, then there is a negative eigenvalue for this problem.

(b) Explain why this makes sense physically given the boundary conditions.

8. (a) Compute the Fourier sine series of $\phi(x) = x$ on $[0, 1]$.

   (b) Integrate the series term by term to find the Fourier cosine series of $\phi(x) = x^2/2$. Find the value of the constant term that arises from integration. This is the first term in the cosine series.

   (c) Explain why we know the cosine series converges pointwise at $x = 0$. What value must it converge to?

   (d) Plug $x = 0$ into your series in part (b) to find the exact value of the sum

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - + \cdots$$

9. (a) Show that the inner product of two functions is a linear operation. ie. Show that

$$(f, c_1 g + c_2 h) \;=\; c_1(f, g) + c_2(f, h)$$

   for any $c_1, c_2 \in \mathbb{R}$ and any integrable functions f, g, h.

   (b) Suppose that $(f, g) = (h, g)$. Does it follow that $f = h$? Prove or provide a counterexample.

10. From scratch, find the complete series solution, with the coefficients, of

$$\begin{aligned} u_t &= 2u_{xx} \quad \text{for } -\pi < x < \pi,\ t > 0 \\ u(-\pi, t) &= u(\pi, t), \\ u_x(-\pi, t) &= u_x(\pi, t), \\ u(x, 0) &= |x| + 1. \end{aligned}$$

    How do we know that there are no negative or complex eigenvalues without doing any tough computations?

11. Use the method of subtraction to solve

$$\begin{aligned} u_{tt} &= 4u_{xx} + 50e^t \sin(5x) \quad \text{for } 0 < x < \pi, \\ u(0, t) &= 0, \\ u(\pi, t) &= 0, \\ u(x, 0) &= 0, \\ u_t(x, 0) &= \sin(3x). \end{aligned}$$

12. Consider the function

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } 1 \le x \le 2 \end{cases}$$

    (a) At what points in the interval $[0, 2]$ does the Fourier sine series for $f(x)$ converge to $f(x)$? (Pay particular attention to the endpoints.)

    (b) At points where the sine series does not converge to $f(x)$, what does it converge to?

    (c) Does the Fourier sine series converge uniformly to $f(x)$ on $[0, 2]$? Can we be sure?

    (d) Does the Fourier sine series converge in the L^2 sense to $f(x)$ on $[0, 2]$? Can we be sure?