MATH 305, Spring 2016

Computer Lab #2

Minimal Surfaces and Soap Film

DUE DATE: Friday, April 1, 5:00 pm

In this project you will learn about an important connection between harmonic functions and minimal surfaces. These surfaces can be realized physically using soap films. You will use Maple to graph these special surfaces and their boundaries. You will also need to perform several calculations by hand in order to understand the theory behind this interesting topic.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, website, another student, etc. should all be appropriately referenced. Please turn in **one report per group**, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, writing in complete sentences. Your report should be TYPED and you are encouraged to type all of it in your Maple worksheet. Calculations can be attached in an appendix or included amidst your report.

Background: Strain Energy and Harmonic Functions

Suppose that you take a piece of wire and bend it into some particular shape, such as a helix. Now dip the wire frame into a bowl of soapy water. What is the surface that will be formed by the soap film containing the wire as its boundary? The thin membrane formed by the soap is an example of a **minimal surface**. It is "minimal" because the membrane is trying to span the wire frame boundary in the easiest, most relaxed fashion. It seeks to minimize the amount of energy used to span the fixed boundary. Of all possible surfaces having the same given boundary (there are an ∞ number of possibilities), the minimal surface is the one with the least energy.

To set up our problem mathematically, let D be some bounded domain in the plane with boundary curve B. Suppose that $u(x, y)$ is a real-valued continuously differentiable function on D whose values on the boundary B match some given real-valued function g. In other words, $u(x, y) = g(x, y)$ for any point $(x, y) \in B$. The graph of g represents the wire frame boundary and the graph of u is a possible membrane stretching across the wire frame. The **energy integral** of u is given by the following double integral:

$$E(u) \;=\; \int\!\!\int_D (u_x^2 + u_y^2)\, dx\, dy.$$

This is sometimes called the **Dirichlet energy**. The quantity $E(u)$ represents the total strain energy for a thin membrane stretched in the shape of the graph of u in $\mathbb{R}^3$. It can be derived using the formula for the surface area of the graph of $u(x, y)$.

We would like to find the function u that agrees with g on the boundary B and has the smallest possible energy integral. This is a calculus problem, but not in the sense you are accustomed to. It is a problem from a field called the **calculus of variations**. We are minimizing $E(u)$ over a space of functions, rather than over an interval or region in the plane.

Goal: Minimize $E(u)$ over all continuously differentiable functions $u(x, y)$ on D satisfying $u = g$ on the boundary B.

There is a surprising connection between this minimization problem and complex analysis. Recall that the real and imaginary parts of an analytic function are each **harmonic**, that is, they are continuously differentiable and satisfy Laplace's equation $u_{xx} + u_{yy} = 0$ on some domain.

Claim: $E(u)$ is minimized by the unique harmonic function on D whose values agree with g on B.

We will verify this claim below, leaving some of the details as exercises. Suppose that u is the function with minimal energy. Let v be any continuously differentiable function on D that equals 0 on the boundary B. The function $u + \epsilon v$, where $\epsilon \in \mathbb{R}$ is some parameter, will also equal g on B, since

$$(u + \epsilon v)(x, y) \;=\; u(x, y) + \epsilon v(x, y) \;=\; g(x, y) + \epsilon \cdot 0 \;=\; g(x, y) \qquad \text{for any } (x, y) \in B.$$

Since u produces the minimal energy integral, we have

$$\begin{aligned} E(u) &\leq E(u + \epsilon v) \\ &= E(u) + 2\epsilon \int\!\!\int_D (u_x v_x + u_y v_y)\, dx\, dy + \epsilon^2 E(v), \end{aligned}$$

from which it follows that

$$2\epsilon \int\!\!\int_D (u_x v_x + u_y v_y)\, dx\, dy + \epsilon^2 E(v) \;\geq\; 0. \tag{1}$$

Since ϵ can either be positive or negative, this implies that

$$\int\!\!\int_D (u_x v_x + u_y v_y)\, dx\, dy \;=\; 0.$$

Next we suppose that u, v, D, and B satisfy the hypotheses of Green's Theorem. Applying Green's Identity, we find that

$$\begin{aligned} 0 &= \int\!\!\int_D (u_x v_x + u_y v_y)\, dx\, dy \\ &= \oint_B v \frac{\partial u}{\partial n}\, ds - \int\!\!\int_D v(u_{xx} + u_{yy})\, dx\, dy \\ &= -\int\!\!\int_D v(u_{xx} + u_{yy})\, dx\, dy. \end{aligned}$$

Here, $\oint_B$ means to do a line integral over the boundary B. Thus, we have shown that

$$\int\!\!\int_D v(u_{xx} + u_{yy})\, dx\, dy \;=\; 0$$

for any smooth function $v(x, y)$ that vanishes on the boundary B. It follows that $u_{xx} + u_{yy} = 0$ on all of D. (If this were not the case, we could create a smooth function v that makes the integral positive, which is a contradiction.) This shows that the unique minimizer of the energy integral must be a harmonic function. QED

Note that the simple function $u = c$ (where c is a fixed constant) is harmonic. The graph of u over any domain D is just a horizontal plane. Thus, any wire frame that is flat (e.g., your typical bubble maker) will produce a simple flat soap film spanning the frame. This is obvious to anyone who has ever made soap bubbles, but now we have a mathematical explanation.

A Simple Example: The Saddle

Throughout the lab, we will always take our domain D to be the open unit disk $|z| < 1$, that is, $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with boundary B equal to the unit circle. Recall that $x = \cos\theta$, $y = \sin\theta$, $0 \leq \theta \leq 2\pi$ parametrizes the unit circle. Suppose that we set $g = \cos(2\theta)$ on the boundary B. Each of the following two-variable functions agree with g on B:

$$u_1(x, y) \ = \ x^2 - y^2, \qquad u_2(x, y) \ = \ 2x^2 - 1, \qquad u_3(x, y) \ = \ 1 - 2y^2.$$

However, only u_1 is a harmonic function. The value of the energy integral $E(u_1)$ will be less than the values for the other two functions. The function u_1 is the real part of $f(z) = z^2$ and its graph is the familiar saddle from multivariable calculus. Consequently, a wire frame in the shape of $\cos(2\theta)$ graphed over the unit circle will form a soap film in the shape of a saddle (see Figures 1 and 2).

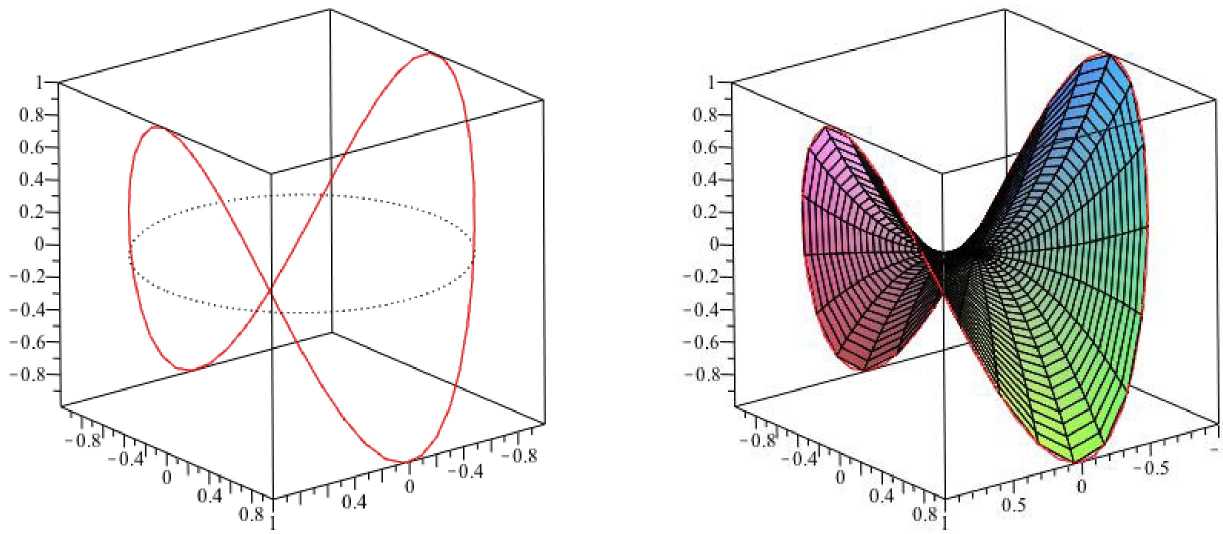

Figure 1: The graph of $g = \cos(2\theta)$ above the unit circle (left) and the graph of the saddle $u_1 = x^2 - y^2$ over D containing the graph of g as its boundary.

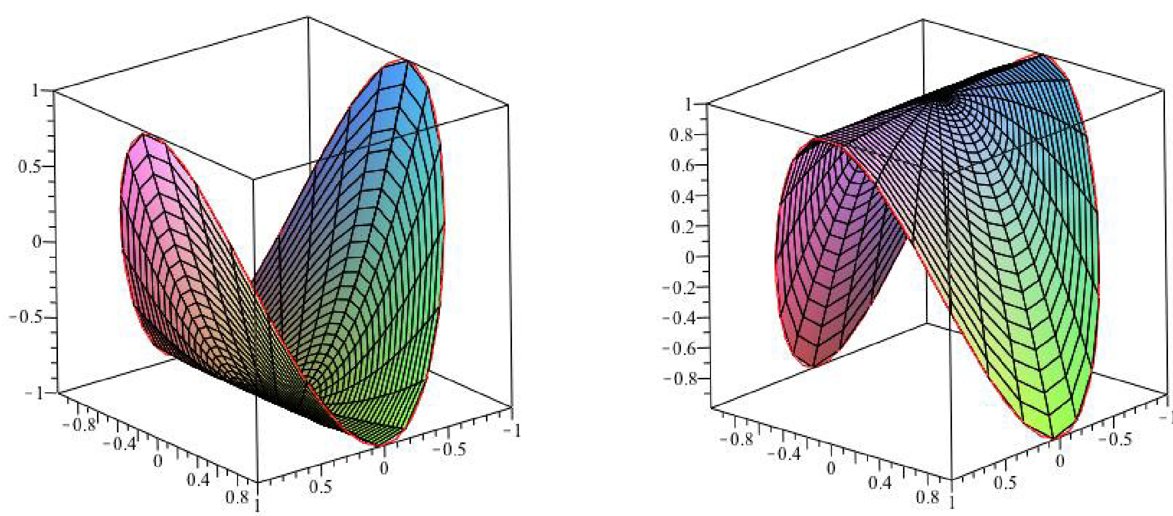

Figure 2: The graph of $u_2 = 2x^2 - 1$ (left) and $u_3 = 1 - 2y^2$ (right) also contain the graph of g as their boundary. However, neither function is harmonic.

The graphs in the previous figures were produced using Maple with some commands in the `plots` package. Start by typing `with(plots):` to load the package. You will need to execute this command every time you start a new Maple session. To draw the graph of the boundary saddle curve, type the command

```
saddleCurve:=spacecurve([cos(t),sin(t),cos(2*t)],t=0..2*Pi,axes=boxed,color=red):
```

which defines a parametric curve in $\mathbb{R}^3$ with $x = \cos t, y = \sin t, z = \cos(2t)$ over the interval $0 \le t \le 2\pi$. The curve has been assigned the name "saddleCurve." In general, to obtain the graph of a different boundary function f, you will need to change z to $z = g(t)$.

To plot the graph of u_1 over the unit disk D, we use cylindrical coordinates. These need to be defined as follows:

```
addcoords(Cylinder,[z,r,theta],[r*cos(theta),r*sin(theta),z]):
```

Note: You can use the variable θ in the above command instead of typing out the full word. To obtain the graph of the saddle, we need to convert $u_1 = x^2 - y^2$ into cylindrical coordinates, which gives $z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$. To plot this function with our newly defined coordinates "Cylinder," run the following command (all on one line in Maple):

```
Saddle:=plot3d(r^2*(cos^2(theta)-sin^2(theta)),r=0..1,theta=0..2*Pi,coords=Cylinder,
   axes=boxed,grid=[28,28]):
```

Notice that the function in cylindrical coordinates is the first entry inside the `plot3d` command. The option `grid=[28,28]` specifies how many rays and circles to plot as the skeleton of the surface. The higher these numbers, the more accurate the plot. Finally, to plot both the boundary curve and the saddle together on the same graph, we use the `display` command.

```
display([Saddle,saddleCurve])
```

Be sure to click on the graph and drag the mouse to see different orientations of the saddle and its boundary curve. You can also carefully alter the 3d-view by clicking on the graph and then adjusting the angles at the top of the screen. For practice, try and produce the graphs in Figure 2 by tweaking the `plot3d` command above. It is advisable to rename your surfaces (e.g., change "Saddle" to "Trough") so you don't get confused as you complete the lab.

Exercises:

1. **(Saddle)** Suppose that $u_1 = x^2 - y^2, u_2 = 2x^2 - 1$ and $u_3 = 1 - 2y^2$.

   a. Show that u_1 is a harmonic function, while u_2 and u_3 are not.

   b. Show that all three functions reduce to $g(\theta) = \cos(2\theta)$ on the unit circle B.

   c. Using polar coordinates, compute the value of the energy integral $E(u)$ over D for each function. Notice that $E(u_1) < E(u_2) = E(u_3)$, as expected.

2. **(Proof)** In this problem you will verify a few steps of the proof that the function minimizing the energy integral must be harmonic.

   a. Using linearity of the integral, show that

$$E(u + \epsilon v) \;=\; E(u) + 2\epsilon \int\!\!\int_D (u_x v_x + u_y v_y)\,dx\,dy + \epsilon^2 E(v).$$

b. Explain why inequality (1) implies that

$$\iint_D (u_x v_x + u_y v_y)\, dx\, dy \;=\; 0.$$

Hint: Suppose that $\epsilon > 0$. Divide both sides of the inequality by ϵ and then let $\epsilon \to 0^+$. Now do it again, but assume that ϵ is negative.

c. In the proof, why is $\oint_B v \frac{\partial u}{\partial n}\, ds = 0$?

Hint: What is the value of the function v on B?

3. **(Cool)** Let $u(r,\theta)$ be the real part of $f(z) = -i\log(z^3)$ in polar coordinates.

a. Find a simple formula for u and show that it satisfies the polar form of Laplace's equation $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$. *Note:* You do not need to worry about a particular branch of the logarithmic function here. We only require that it satisfies Laplace's equation "locally."

b. What is the image of the unit circle under u? What shape is the graph of u over the unit disc D?

c. Use Maple to plot the surface and the boundary curve together over the domain $0 < r \leq 1, 0 \leq \theta \leq 4\pi$. You will need to run the command

```
addcoords(NewCylinder,[z,theta,r],[r*cos(theta),r*sin(theta),z])
```

to define your coordinates. Then set `coords=NewCylinder` in your `plot3d` command.

4. **(Hard)** Suppose that $f(z) = \dfrac{1}{\pi}\left(z - \dfrac{1}{z}\right) \operatorname{Log}\left(\dfrac{1-z}{1+z}\right)$ where $\operatorname{Log} z$ is the principal branch of the logarithmic function. Using Taylor series (a coming attraction), one can show that $z = 0$ is a removable singularity and that f is analytic on the open unit disk D.

a. Setting $z = re^{i\theta}$, show that the real part of f over D is

$$u(r,\theta) = \frac{1}{\pi}\left[\frac{1}{2}\left(r - \frac{1}{r}\right) \cdot \cos\theta \cdot \ln\left(\frac{r^2 - 2r\cos\theta + 1}{r^2 + 2r\cos\theta + 1}\right) + \left(r + \frac{1}{r}\right) \cdot \sin\theta \cdot \tan^{-1}\left(\frac{2r\sin\theta}{1 - r^2}\right)\right].$$

b. By computing $\operatorname{Re}(f(e^{i\theta}))$, show that the image of the unit circle under u is given by $g(\theta) = |\sin\theta|$. You need to consider two cases: $\sin\theta > 0$ and $\sin\theta < 0$.

c. Use Maple to plot the boundary curve $g(\theta) = |\sin\theta|$ along with the graph of $u(r,\theta)$ over the unit disk. You should use cylindrical coordinates, defining them via

```
addcoords(Cylinder,[z,r,theta],[r*cos(theta),r*sin(theta),z]):
```

5. **(Fun)** Create your own interesting function with associated boundary curve. State your analytic function $f(z)$, its real part u and the boundary function g over the unit circle. Use Maple to make a plot of your surface and boundary together. For bonus points, find some wire and make a frame in the shape of your boundary curve. Then we will dip it in a solution of soapy water in class to see how well the soap film matches your graph of u.

References:

1. Fisher, Stephen D., *Complex Variables*, second ed., Dover Publications Inc., 1999, pp. 245–254.

2. Sethian, J. A., Minimal Surfaces and Soap Bubbles, accessed March 18, 2016, `https://math.berkeley.edu/~sethian/2006/Applications/Minimal_Surfaces/minimal.html`

3. Weisstein, Eric W., Minimal Surface, *MathWorld–A Wolfram Web Resource*, accessed March 20, 2016, `http://mathworld.wolfram.com/MinimalSurface.html`