MATH 305 Complex Analysis Spring 2016

Sample Final Exam Questions

1. Express each quantity in rectangular form $a + ib$ and polar form $re^{i\theta}$.

   (a) $\dfrac{-8i}{\sqrt{3}+i}$

   (b) $(1-i)^{12}$

2. Prove algebraically that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, \quad (z_2 \neq 0).$

3. Find the four roots $(-8 - 8\sqrt{3}\,i)^{1/4}$ in polar and rectangular form, and sketch the square in the complex plane that has these roots as its vertices.

4. Let R be the region in the complex plane defined by

$$R = \{z \mid -1 \leq \mathrm{Re}(z) \leq 1, -\pi/2 \leq \mathrm{Im}(z) \leq \pi/2\}$$

   (a) Sketch the region R in the complex plane.

   (b) Sketch the image of R under the map $f(z) = e^z$.

   (c) Sketch the image of R under the map $g(z) = e^{2z} = (e^z)^2$.

5. Evaluate the following limits, if they exist. Show your work, making sure to justify your answers thoroughly.

   (a) $\displaystyle\lim_{z\to\infty} \frac{iz^4 - 3z^2}{(4-3iz^2)^2}$

   (b) $\displaystyle\lim_{z\to 0} \left(\frac{\overline{z}}{z}\right)^2$

   (c) $\displaystyle\lim_{z\to -2i} \frac{z^{10} - 50}{(z^2+4)^2}$

6. Use the SDC theorem to determine where each of the following functions is differentiable. Give a formula for the derivative whererever the function is differentiable.

   (a) $f(z) = ze^{iy}$

   (b) $f(z) = -3x^2y + y^3 - x^2 + y^2 - y + i(x^3 - 3xy^2 + x - 2xy)$

7. Find a function $v(x, y)$ such that the function $f(z) = (y + e^x \sin y) + i\,v(x, y)$ is entire.

8. Use parametrizations to find $\displaystyle\int_{C_i} (\overline{z})^2\, dz$ where

   (a) C_1 is the line segment from i to $-i$,

   (b) C_2 is the portion of the unit circle from $-i$ to i, traversed counterclockwise,

   (c) $C_3 = C_1 + C_2$ is the combination of the first two contours.

   (d) Why isn't the integral in part **(c)** equal to 0?

9. Let C be the square with vertices -2, 2, $2+3i$, and $-2+3i$, traversed in the counterclockwise direction. For each function $f(z)$ below, compute $\oint_C f(z)\,dz$. Be sure to specify what theorem or formula you are using.

   (a) $f(z) = \dfrac{\sin z}{z - \pi/2 + i}$

   (b) $f(z) = \dfrac{\sin z}{z - (\pi/2 + i)}$

   (c) $f(z) = \dfrac{z^3}{[z - (1+i)]^3}$

   (d) $f(z) = \dfrac{1}{z^4 + 5z^2 + 4}$

10. Suppose that f and g are entire functions and that $f(z) = g(1/z)$ for all $z \in \mathbb{C} - \{0\}$. Use Liouville's theorem to prove that f must be constant.

11. Let

$$f(z) = \frac{1}{z(z-3)}.$$

   (a) Using $\dfrac{1}{z(z-3)} = \dfrac{1}{z} \cdot \dfrac{-1}{3(1 - z/3)}$, find the Laurent series expansion of $f(z)$ in the punctured disk $0 < |z| < 3$. Give the first four terms in the series as well as a closed form expression for the series.

   (b) Find $\operatorname*{Res}_{z=0} f(z)$ and $\operatorname*{Res}_{z=3} f(z)$.

   (c) Explain why $\oint_C f(z)\,dz = 0$ for any simple closed contour C that encloses both 0 and 3.

12. Suppose that $g(z) = z^3 \cos(1/z)$.

   (a) Find the Laurent series expansion for $g(z)$ about $z_0 = 0$. For what points in $\mathbb{C}$ does it converge?

   (b) What type of singular point is $z_0 = 0$?

   (c) What is the residue of $g(z)$ at $z_0 = 0$?

13. Find the value of the integral

$$\oint_C \frac{3z^3 + 2}{(z-1)(z^2+9)}\,dz\,,$$

   where

   (a) C is the circle $|z - 2| = 2$ taken counterclockwise,

   (b) C is the circle $|z| = 4$ taken clockwise.

14. Use residues to compute the real improper integral

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx\,.$$

   Be sure to justify all the steps taken in your calculation.

15. **Complex Analysis Potpourri:**

(a) If the Cauchy-Riemann equations are satisfied for the function $f(z)$ at the point z_0, then $f'(z_0)$ exists. This statement is (choose one): TRUE FALSE

(b) State the principal branch of the function $\mathrm{Log}(z)$ and sketch the domain of the function. Explain why the branch cut is necessary.

(c) Find and simplify the principal value of $(4i)^{-i}$.

(d) What are the singular points of $g(z) = \tan z$?

(e) If $f(z) = \dfrac{1}{e^z + 1}$, what is the radius of the largest disk about $z_0 = 0$ for which the Taylor series of $f(z)$ is guaranteed to converge?

16. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, explain why or provide a **counterexample**.

(a) $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ for any $z_1, z_2 \in \mathbb{C}$, where $\mathrm{Log}\, z$ is the principal value of the logarithmic function.

(b) If $f(z)$ is continuous on a domain D and $\oint_C f(z)\, dz = 0$ for every closed contour C in D, then $f(z)$ is analytic in D.

(c) The sum of the infinite series $1 + i + i^2 + i^3 + i^4 + \cdots$ is $\dfrac{1+i}{2}$.

(d) $\lim_{z \to \infty} e^z = \infty$.

(e) The function $g(z) = \dfrac{e^z - 1 - z}{z^2}$ has a removable singularity at $z = 0$.