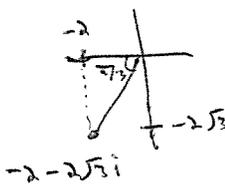


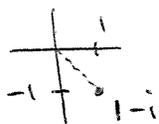
Solutions to Sample Final Exam Questions

1. (a) $\frac{-8i}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-8\sqrt{3}i - 8}{4} = -2\sqrt{3}i - 2 = -2 - 2\sqrt{3}i$
 $= 4e^{i\frac{4\pi}{3}}$

$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = 4$



(b) $1-i = \sqrt{2} \cdot e^{-i\frac{\pi}{4}}$
 so $(1-i)^{12} = (\sqrt{2}e^{-i\frac{\pi}{4}})^{12} = 2^6 \cdot e^{-i3\pi} = -64$



2. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
 $\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$
 $= \frac{x_1x_2 + y_1y_2 + i(-x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}$ so $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)}{x_2^2 + y_2^2}$

Then, $\frac{\overline{z_1}}{\overline{z_2}} = \frac{x_1 - iy_1}{x_2 - iy_2} \cdot \frac{(x_2 + iy_2)}{(x_2 + iy_2)} = \frac{x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)}{x_2^2 + y_2^2} = \overline{\left(\frac{z_1}{z_2}\right)}$

3. $z = -8 - 8\sqrt{3}i = 16e^{i\frac{4\pi}{3}}$ so $z^{\frac{1}{4}} = 2e^{i(\frac{\pi}{3} + \frac{2\pi}{4}k)}$, $k = \{0, 1, 2, 3\}$

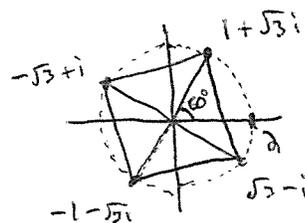
$r = \sqrt{(-8)^2 + (-8\sqrt{3})^2}$
 $= \sqrt{8^2 + 3 \cdot 8^2}$
 $= \sqrt{4 \cdot 8^2} = 2 \cdot 8 = 16$

$= 2e^{i\frac{\pi}{3}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{4\pi}{3}}, 2e^{i\frac{11\pi}{6}}$

$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right), 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right),$

$2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

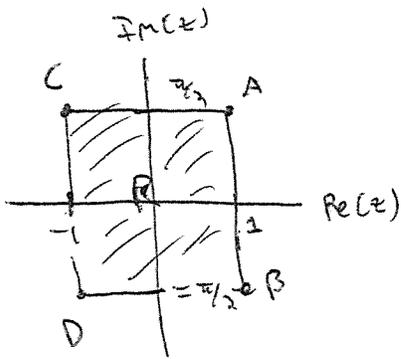
$= \pm(1 + \sqrt{3}i), \pm(-\sqrt{3} + i)$



The roots lie on a square of radius 2 rotated 60° off pos. x-axis.

4.

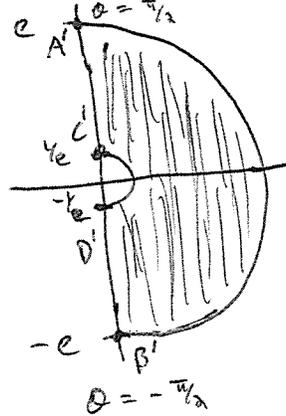
(a)



(b) $e^z = e^{x+iy} = e^x \cdot e^{iy}$

modulus ranges from e^{-1} to e while

argument θ from $-\pi/2$ to $\pi/2$



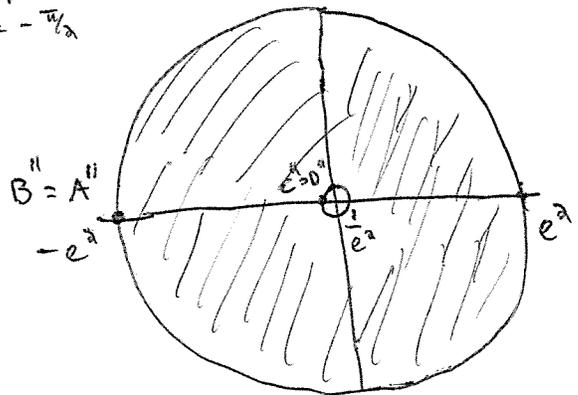
(c) $z^2 = (e^z)^2 = e^{2x} \cdot e^{i2y}$

square the modulus and double the angle.

modulus now ranges from e^{-2} to e^2 while

argument θ from $-\pi$ to π

\Rightarrow image of R under g is annulus with $\frac{1}{e^2} < r < e^2$.



5. (a) $\frac{-i}{9}$ use L'Hôpital: $\lim_{z \rightarrow 0} \frac{i(\frac{1}{z})^4 - 3(\frac{1}{z})^2}{(4 - 3i(\frac{1}{z})^2)^2} = \frac{\frac{i}{z^4} - \frac{3}{z^2}}{(4 - \frac{3i}{z^2})^2}$

$= \lim_{z \rightarrow 0} \frac{i - 3z^2}{(4z^2 - 3i)^2} = \frac{i - 0}{(0 - 3i)^2} = \frac{i}{-9}$

(b) D.M.E. Let $z = x$. $\lim_{x \rightarrow 0} \left(\frac{x}{x}\right)^2 = \lim_{x \rightarrow 0} (1)^2 = 1$

Let $z = x + ix \Rightarrow \lim_{x \rightarrow 0} \left(\frac{x-ix}{x+ix}\right)^2 = \lim_{x \rightarrow 0} \left(\frac{1-i}{1+i}\right)^2 = \lim_{x \rightarrow 0} \frac{-2i}{2i} = -1$

Since $1 \neq -1$, we have two different limits coming from different directions. $\therefore \lim_{z \rightarrow 0} \left(\frac{z}{z}\right)^2$ does not exist.

(c) ∞ use L'Hôpital: $\lim_{z \rightarrow -2i} \frac{1}{\frac{z^{10}-50}{(z^2+4)^2}} = \lim_{z \rightarrow -2i} \frac{(z^2+4)^2}{z^{10}-50} = \frac{0}{-1624-50} = \frac{0}{-1674-50} = 0$.

\therefore Since $\lim_{z \rightarrow -2i} \frac{1}{g(z)} = 0$, $\lim_{z \rightarrow -2i} f(z) = \infty$.

6. (a) $f(z) = ze^{iy} = (x+iy)e^{iy} = (x+iy)(\cos y + i \sin y)$

$$= \underbrace{x \cos y - y \sin y}_u + i \underbrace{(y \cos y + x \sin y)}_v$$

$$u_x = \cos y \quad v_y = \cos y - y \sin y + x \cos y$$

So $u_x = v_y$ iff $-y \sin y + x \cos y = 0$

$$u_y = -x \sin y - \sin y - y \cos y \quad v_x = \sin y$$

So $u_y = -v_x$ iff $x \sin y + y \cos y = 0$

But $(-y \sin y + x \cos y = 0) \times$ $\Rightarrow \cos y (x^2 + y^2) = 0$ so $x=y=0$ or $\cos y = 0$
 $+ (x \sin y + y \cos y = 0) \times$

and

$$(-y \sin y + x \cos y = 0) \times y \Rightarrow \sin y (x^2 + y^2) = 0 \Rightarrow x=y=0 \text{ or } \sin y = 0$$

$$+ (x \sin y + y \cos y = 0) \times x$$

But $\cos y = \sin y = 0$ is impossible b/c $\cos^2 y + \sin^2 y = 1 \neq 0$.

$\therefore f$ is differentiable only at $(0,0)$. (SCD theorem conditions are satisfied since partials exist and are continuous at $(0,0)$.)

$$f'(0) = u_x(0,0) + i v_x(0,0)$$

$$= 1 + i \cdot 0$$

$$= 1.$$

$$6(b). \quad u = -3x^2y + y^3 - x^2 + y^2 - y, \quad v = x^3 - 3xy^2 + x - 2xy$$

$$u_x = -6xy - 2x = v_y$$

$$u_y = -3x^2 + 3y^2 + 2y - 1 = -v_x$$

so CR-equations are satisfied on all of \mathbb{C} .

By SCD THM., since partials exist and are continuous on all of \mathbb{C} , $f(z)$ is analytic on all of \mathbb{C} (f is an entire function).

$$7. \quad \text{Let } u(x,y) = y + e^x \sin y.$$

$$u_x = e^x \sin y \Rightarrow u_{xx} = e^x \sin y$$

$$u_y = 1 + e^x \cos y \Rightarrow u_{yy} = -e^x \sin y$$

$\Rightarrow u_{xx} + u_{yy} = 0$ so u is harmonic.

Find a harmonic conjugate v .

$$u_x = e^x \sin y = v_y \Rightarrow v = -e^x \cos y + \phi(x)$$

$$\Rightarrow v_x = -e^x \cos y + \phi'(x)$$

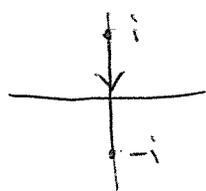
$$\text{Put } v_x = -u_y = -e^x \cos y - 1 \Rightarrow -e^x \cos y + \phi'(x) = -e^x \cos y - 1$$

$$\text{so } \phi'(x) = -1 \Rightarrow \phi(x) = -x + c, \quad c \in \mathbb{R}.$$

$\therefore v(x,y) = -e^x \cos y - x + c$ will make $f = u + iv$ an entire function.

Note: If $c=0$, $f(z) = -i(z + e^z)$.

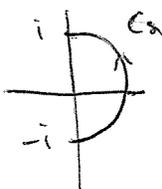
$$8. (a) \quad C_1: z = (1-t)i + t(-i) = i - 2it = i(1-2t), \quad 0 \leq t \leq 1$$



(or Exam #2)

$$\begin{aligned} \int_{C_1} (\bar{z})^2 dz &= \int_0^1 (i(2t-1))^2 \cdot -2i dt = +2i \int_0^1 (2t-1)^2 dt \\ &= 2i \frac{(2t-1)^3}{3} \cdot \frac{1}{2} \Big|_0^1 = \frac{i}{3} (1 - -1) = \frac{2i}{3} \end{aligned}$$

8(b). $C_2: z = e^{i\theta}, -\pi/2 \leq \theta \leq \pi/2$



$$\int_{C_2} (z)^2 dz = \int_{-\pi/2}^{\pi/2} (e^{-i\theta})^2 \cdot i e^{i\theta} d\theta = i \int_{-\pi/2}^{\pi/2} e^{-i\theta} d\theta$$

$$= i \cdot \left. \frac{-1}{i} e^{-i\theta} \right|_{-\pi/2}^{\pi/2} = - (e^{-i\pi/2} - e^{i\pi/2}) = -(-i - i) = 2i$$

8(c): $C_3 = C_1 + C_2$. By linearity, $\int_{C_3} (z)^2 dz = \int_{C_1} (z)^2 dz + \int_{C_2} (z)^2 dz$

8(d): The Cauchy-Goursat theorem does not apply b/c $(\bar{z})^2$ is not an analytic function. $= \frac{2}{3}i + 2i = \frac{8i}{3}$

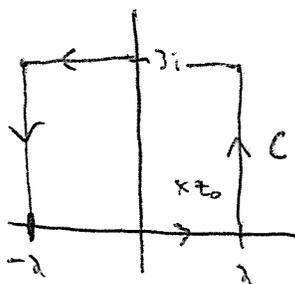
$$(\bar{z})^2 = (x-iy)^2 = x^2 - y^2 - 2xyi$$

↳ nowhere analytic.

It also does not have an anti-deriv. so the AD Thm. does not apply either.

$$\begin{aligned} u_x &= 2x & v_y &= -2x \\ &\Rightarrow x=0 \\ u_y &= -2y & -v_x &= 2y \\ &\Rightarrow y=0. \end{aligned}$$

9.



(a) $f(z) = \frac{\sin z}{z - \pi/2 + i}$

↳ outside C
 $z_0 = \pi/2 - i$

$\sin z$ entire function
 $\Rightarrow f$ is entire so

$\oint_C f(z) dz = 0$ by Cauchy-Goursat Thm.

(b) $f(z) = \frac{\sin z}{z - (\pi/2 + i)}$ $z_0 = \pi/2 + i$ is inside C
 $Q(z) = \sin z$

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i \cdot Q(z_0) \text{ by Cauchy Integral formula} \\ &= 2\pi i \cdot \sin(\pi/2 + i) = 2\pi i \cdot \frac{1}{2i} \left(e^{(\pi/2+i)i} - e^{-i(\pi/2+i)} \right) \\ &= \pi \left[e^{-1+\pi/2 i} - e^{1-\pi/2 i} \right] = \pi \left[e^{-1} \cdot i - e^1 \cdot (-i) \right] = i\pi \left(\frac{1}{e} + e \right) \end{aligned}$$

9(c). $f(z) = \frac{z^3}{(z-(1+i))^3}$

$z_0 = 1+i$ is inside C , pole of order 3 ($m=3$)
use the extension of Cauchy Integral formula
or Cauchy residue theorem.

$Q(z) = z^3$

$Q'(z) = 3z^2$

$Q''(z) = 6z$

$$\int_C f(z) dz = 2\pi i \cdot \frac{Q''(1+i)}{2!} = \pi i (6(1+i)) = \pi i (6+6i) = -6\pi + 6\pi i$$

9(d). $f(z) = \frac{1}{z^4 + 5z^2 + 4} = \frac{1}{(z^2+1)(z^2+4)}$
 $= \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$

so $z_0 = i, z_1 = 2i$
are inside C . (both simple poles)

By the Cauchy residue theorem, $\int_C f(z) dz = 2\pi i [\text{Res}_{z=i} f(z) + \text{Res}_{z=2i} f(z)]$

$$= 2\pi i \left[\frac{1}{2i \cdot 3i \cdot -i} + \frac{1}{3i \cdot -i \cdot 4i} \right] = 2\pi i \left[\frac{1}{6i} + \frac{1}{-12i} \right] = 2\pi \left[\frac{1}{12} \right]$$

$Q_0(z) = \frac{1}{(z+i)(z+2i)(z-2i)}$ \rightarrow $Q_1(z) = \frac{1}{(z+i)(z-i)(z+2i)} = \frac{\pi}{6}$

10. Consider $\lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} g\left(\frac{1}{z}\right) \stackrel{\text{LIP I (2)}}{=} \lim_{z \rightarrow 0} g(z) = g(0)$ b/c g is continuous on \mathbb{C} .

$\therefore \exists R > 0$ s.t. $|f(z)| \leq |g(0)| + 1 \quad \forall z : |z| > R$
(f is bounded near ∞)

Since f is continuous on $|z| \leq R$, $\exists M > 0$ s.t. $|f(z)| \leq M$

$\forall z : |z| \leq R$. (continuous functions on compact sets have a Max)

$\therefore |f(z)| \leq \max \{ M, |g(0)| + 1 \} \Rightarrow f$ is bounded $\forall z \in \mathbb{C}$

By Liouville's Thm., since f is entire + bounded, it is constant.

11. (a) $\frac{1}{z(z-3)} = \frac{1}{z} \cdot \frac{-1}{3} \cdot \frac{1}{1 - \frac{z}{3}}$ \nearrow geometric series with $r = \frac{z}{3}$

$$= -\frac{1}{3z} \left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \dots \right) \text{ for } \left|\frac{z}{3}\right| < 1, z \neq 0$$

$$= -\frac{1}{3z} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots \right) \text{ for } |z| < 3, z \neq 0$$

$$= -\frac{1}{3z} - \frac{1}{9} - \frac{1}{27}z - \frac{1}{81}z^2 - \dots \text{ for } 0 < |z| < 3$$

$$= -\frac{1}{3z} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n \text{ for } 0 < |z| < 3.$$

(b) using the series above, we have $\text{Res}_{z=0} f(z) = -\frac{1}{3}$ b/c this is the coefficient of $\frac{1}{z} = \frac{1}{z-0}$ term.

Since $\frac{1}{z(z-3)} = \frac{\frac{1}{3}z}{z-3}$, $\text{Res}_{z=3} f(z) = \frac{1}{3}$ (evaluate $Q(z) = \frac{1}{3}$ at $z=3$).

\downarrow
simple pole

Or, use partial fractions: $\frac{1}{z(z-3)} = \frac{-1/3}{z} + \frac{1/3}{z-3}$.

(c) Since $\text{Res}_{z=0} f(z) + \text{Res}_{z=3} f(z) = \frac{1}{3} - \frac{1}{3} = 0$, by the Cauchy residue theorem, $\oint_C f(z) dz = 2\pi i \{ \text{residues} \} = 2\pi i \cdot 0 = 0$.

12. (a) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$ converges $\forall z \in \mathbb{C}$

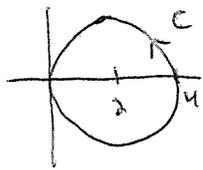
$\Rightarrow \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \dots$ $\forall z \in \mathbb{C} - \{0\}$

$\Rightarrow z^3 \cos\left(\frac{1}{z}\right) = z^3 - \frac{1}{2}z + \frac{1}{24z} - \frac{1}{6!z^3} + \dots = z^3 - \frac{1}{2}z + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+4)!} z^{3-2n}$

(b) essential singularity, b/c there are an ∞ # of terms in principal part

(c) $\text{Res}_{z=0} g(z) = \frac{1}{24}$

13. (a) $|z-1| = 2$



$$f(z) = \frac{3z^3 + \lambda}{(z-1)(z^2+9)}$$

$z_0 = 1$ is only singular point inside C

Res $f(z) = Q(z)$ where $z=1$

$$Q(z) = \frac{3z^3 + \lambda}{z^2 + 9} \quad \rightarrow \quad \frac{5}{10} = \frac{1}{2}$$

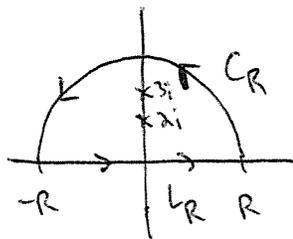
$$\therefore \oint_C f(z) dz = 2\pi i \cdot \frac{1}{2} = \pi i$$

(b) $|z|=4$ has three singular points inside it: $1, 3i$ and $-3i$, all simple poles

$$\text{Res}_{z=3i} f(z) = \frac{3(3i)^3 + \lambda}{(3i-1)(3i+3i)} = \frac{-81i + \lambda}{-18 - 6i}, \quad \text{Res}_{z=-3i} f(z) = \frac{3(-3i)^3 + \lambda}{(-3i-1)(-3i-3i)} = \frac{81i + \lambda}{-18 + 6i}$$

$$\therefore \oint_C f(z) dz = 2\pi i \cdot \left[\frac{1}{2} + \frac{-81i + \lambda}{-18 - 6i} + \frac{81i + \lambda}{-18 + 6i} \right] = 6\pi i$$

14. $\frac{\pi}{200}$ Let $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$ and let $C = C_R + L_R$. $R > 3$



$$\oint_C f(z) dz = 2\pi i \left[\text{Res}_{z=2i} f(z) + \text{Res}_{z=3i} f(z) \right]$$

$$\int_{-R}^R \frac{x^2}{(x^2+9)(x^2+4)^2} dx + \int_{C_R} f(z) dz$$

$$\text{So } \frac{\pi}{200} \cdot \{ \text{residues} \} = \int_0^R \frac{x^2}{(x^2+9)(x^2+4)^2} dx + \frac{1}{2} \int_{C_R} f(z) dz$$

(i) Find residues: $z=3i$ is a simple pole, so $\text{Res}_{z=3i} f(z) = \frac{(3i)^2}{6i \cdot 25} = \frac{3}{50}i$

$z=2i$ is a pole of order 2.

$$Q(z) = \frac{z^2}{(z^2+9)(z+2i)^2} = \frac{z^2}{z^3 + 2iz^2 + 9z + 18i}$$

Find $Q'(2i)$.

14. Quotient Rule:

$$Q'(z) = \frac{(z^2+9)(z+2i)^3 \cdot \lambda z - z^2 [2z(z+2i)^2 + (z^2+9) \lambda (z+2i)]}{(z^2+9)^2 (z+2i)^4}$$

$$Q'(2i) = \frac{5(4i)^3 \cdot 4i + 4 [4i \cdot 16 + 5 \cdot 2 \cdot 4i]}{25 \cdot 4^2} \dots = \frac{-13i}{200}$$

$$\therefore \pi i \cdot \sum \text{residues} = \pi i \left[\frac{3}{50} i - \frac{13}{200} i \right] = -\pi \left[\frac{12-13}{200} \right] = \frac{\pi}{200}$$

Next, $|f(z)| \leq \frac{R^2}{(R^2-9)(R^2-4)^2}$ on C_R using $||z_1 - z_2|| \leq |z_1 + z_2|$

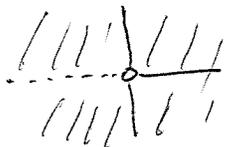
So, by "ML-Theorem", $\left| \int_{C_R} f(z) dz \right| \leq \frac{R^2}{(R^2-9)(R^2-4)^2} \cdot \pi R$

which goes to 0 as $R \rightarrow \infty$.

$$\therefore \frac{\pi}{200} = \int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx$$

15. (a) False. The partial derivatives must also be continuous at z_0 .

(b) $\text{Log}(z) = \ln r + i\theta$ with $-\pi < \theta \leq \pi$,
 $z = re^{i\theta}$.
 Domain is $\mathbb{C} - \{x \leq 0, y = 0\}$.



Required b/c otherwise function is not continuous on neg. real axis.

(c) $(4i)^{-i} = e^{-i \text{Log}(4i)} = e^{-i(\ln 4 + i \cdot \frac{\pi}{2})} = e^{-i \ln 4} \cdot e^{\frac{\pi}{2}}$
 $= e^{\frac{\pi}{2}} (\cos(\ln 4) - i \sin(\ln 4))$

15 (d). $\tan z = \frac{\sin z}{\cos z}$ so the singular points of $\tan z$ are the roots of $\cos z$.

Recall that $|\cos z|^2 = \cos^2 x + \sinh^2 y$, so $\cos z = 0$ iff

$$\cos^2 x + \sinh^2 y = 0 \text{ iff } \cos x = 0, \sinh y = 0$$

$$\text{iff } x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \text{ and } y = 0 \text{ (since } \sinh y = \frac{e^y - e^{-y}}{2} \text{)}$$

\therefore Roots of $\cos z$ are all real, and are precisely the roots of $\cos x$.

$$z = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$$

(e) $f(z) = \frac{1}{z+1}$ is analytic except for when $e^z = -1$ or $z = i\pi + 2n\pi i$, $n \in \mathbb{Z}$

The point $z_0 = i\pi$ is the closest "bad" point to the origin,

so $R = \pi$ is the largest radius of convergence b/c $|i\pi| = \pi$.

16. (a) False. $z_1 = z_2 = -1$ gives $\log(z_1) + \log(z_2) =$
 $i\pi + i\pi = 2\pi i$

but $\log(z_1 z_2) = \log(1) = 0$. $2\pi i \neq 0$.

(b) True. (see p. 169)

As Thm. implies that $f(z)$ has an antiderivative $F(z)$ in D .

$\therefore F'(z) = f(z)$ so that f is the derivative of an analytic function. Since analytic functions have analytic derivatives, f must also be analytic.

(c) False. The series diverges b/c $r = i$ has $|r| = 1$, so it is a divergent geometric series.

(d) False. The limit does not exist. ∞ is an essential singularity for e^z

If $y > 0$ and $x > 0$, $\lim_{x \rightarrow \infty} e^x = \infty$ but if $x < 0$, $\lim_{x \rightarrow \infty} e^{-x} = 0$.

or use Picard's Theorem.