# MATH 305 Complex Analysis

**Sample Questions for Exam 2**

1. Which of the following functions are entire, that is, analytic on the entire complex plane? Provide justification.

   **(a)** $f(z) = e^{-z^2}$

   **(b)** $f(z) = e^{-y}\sin x - ie^{-y}\cos x$

   **(c)** $f(z) = \dfrac{2z+3}{z^2+8}$

2. Show that $u(x, y) = xy + e^{-2y}\cos(2x)$ is a harmonic function and find a harmonic conjugate $v(x, y)$.

3. Find and simplify the principal value of each of the following:

   **(a)** $(-i)^{1+2i}$

   **(b)** $\sin(\pi + i)$

   **(c)** $\mathrm{Log}(-3\sqrt{3} + 3i)$

4. Suppose that the branch $\log z = \ln r + i\,\theta$ $(r > 0, \frac{3\pi}{2} < \theta < \frac{7\pi}{2})$ is specified for the logarithmic function.

   **(a)** Compute $\log(2 + 2i)$.

   **(b)** True or False: $\log(i^2) = 2\log(i)$.

5. Compute the following contour integrals – use parametrizations for the first three. Simplify your answers.

   **(a)** $\displaystyle\int_C \bar{z}\,dz$ where $C$ is the line segment from 1 to $i$.

   **(b)** $\displaystyle\oint_C \frac{1}{z}\,dz$, where $C$ is the unit circle, traversed clockwise.

   **(c)** $\displaystyle\oint_C \frac{1}{z^2}\,dz$, where $C$ is the unit circle, traversed counterclockwise.

   **(d)** $\displaystyle\int_i^{i+2} ze^{z^2}\,dz$

6. Let $C$ be the square with vertices $2 + 2i$, $-2 + 2i$, $-2 - 2i$ and $2 - 2i$, traversed in the counterclockwise direction. For each function $f(z)$ below, compute $\displaystyle\oint_C f(z)\,dz$. Be sure to specify what theorem or formula you are using.

   **(a)** $f(z) = \dfrac{e^z}{z - (1 + \frac{1}{2}\pi i)}$

   **(b)** $f(z) = \dfrac{e^z}{z - (2 + 3i)}$

   **(c)** $f(z) = \dfrac{\cos z}{(z + i)(z^2 + 9)}$

7. Without computing the integral, show that

$$\left| \oint_C (e^{iz} - z^2)\, dz \right| \le 72,$$

where $C$ is the square with vertices $0$, $2$, $2+2i$ and $2i$, traversed in the counterclockwise direction.

8. Let $C$ be the unit circle $z = e^{i\theta}$, $-\pi \le \theta \le \pi$.

   **(a)** Show that for any real constant $a$, $\displaystyle\oint_C \frac{e^{az}}{z}\, dz = 2\pi i$.

   **(b)** By converting the integral in part **(a)** into $\theta$ and $d\theta$, derive the formula

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta \;=\; \pi.$$

9. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, explain why or provide a **counterexample**.

   **(a)** $\operatorname{Log}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Log}(z_1) - \operatorname{Log}(z_2)$ for any $z_1, z_2 \in \mathbb{C}$.

   **(b)** $e^{-iz} = \cos z - i \sin z$ for any $z \in \mathbb{C}$.

   **(c)** $g(z) = e^{\cos z} \cdot \sin z$, $z \in \mathbb{C}$ is an odd function, that is $g(-z) = -g(z)$.

   **(d)** $z^{c_1} z^{c_2} = z^{c_1 + c_2}$ for any complex numbers $z, c_1, c_2$, where all powers are taken to be the principal values.

   **(e)** $\displaystyle\oint_C \frac{-1}{(z-1)^{2017}}\, dz = 0$ for any simple closed contour $C$ not passing through $z = 1$.