MATH 305 Complex Analysis

Sample Questions for Exam 1

1. Express each quantity in rectangular form $x + iy$ and polar form $re^{i\theta}$.

   (a) $\dfrac{1+2i}{3-4i}$

   (b) $(1+\sqrt{3}\,i)^8$

2. Prove that $|z_1 z_2| = |z_1||z_2|$ in two different ways:

   (a) using rectangular coordinates $x + iy$,

   (b) using the property $|z|^2 = z\overline{z}$.

3. **(a)** Draw a figure and explain geometrically why $|z+i| + |z-i| \geq 2$ for all $z \in \mathbb{C}$. For which z does equality hold?

   (b) Sketch and describe the set of all points satisfying $|z+i| + |z-i| = 4$.

4. Write the cube roots of $-8i$ in polar and rectangular form, and sketch the triangle in the complex plane that has these roots as its vertices.

5. Let R be the region in the complex plane defined by

$$R = \{z \mid 0 \leq \mathrm{Re}(z) \leq 2\}$$

   (a) Sketch the region R in the complex plane.

   (b) Sketch the image of R under the map $g(z) = e^z$.

6. Use the ϵ-δ definition of the limit to prove that $\lim_{z \to 3i} 2z^2 - 4iz = -6$.

   Hint: You will need to use the triangle inequality as well as the inequality $|z_1+z_2| \geq ||z_1|-|z_2||$ in your proof. You should end up choosing $\delta = \min\{1, \epsilon/10\}$.

7. Let $f(z) = 1/z$.

   (a) Write f in the form $u(x,y) + iv(x,y)$ and in the form $u(r,\theta) + iv(r,\theta)$.

   (b) Verify that the Cauchy-Riemann equations hold in both rectangular form and in polar form.

   (c) Show that $f'(z) = -1/z^2$.

   (d) What is the image of the unit circle under f?

   (e) What is the image of the circle $|z| = r$ under f?

   (f) What is the image of the ray $\theta = \pi/4, 0 < r \leq 1$ under f?

8. Evaluate the following limits, if they exist. Show your work, making sure to justify your answers thoroughly.

   (a) $\displaystyle\lim_{z \to \infty} \frac{z^2}{(iz+1)^2}$

   (b) $\displaystyle\lim_{z \to 2i} \frac{z}{z^2+4}$

9. Use the SDC theorem to determine where each of the following functions is differentiable and then compute $f'(z)$ at these points.

   (a) $f(z) = y^2 + ix^2$

   (b) $f(z) = x - 2xy + i(x^2 + y - y^2)$

10. Consider the function f defined as

$$f(z) = \begin{cases} \dfrac{z^3}{(\overline{z})^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

   (a) Show that f is continuous at $z = 0$.

   (b) Show that f is not differentiable at $z = 0$.