

1. (a) $f(z) = e^{-z^2}$ is entire b/c it is the composition of two entire functions, e^z and $-z^2$.

$$(b) f(z) = e^{-y} \sin x - i e^{-y} \cos x \text{ has } u(x,y) = e^{-y} \sin x \\ v(x,y) = -e^{-y} \cos x$$

(R-equations:

$$u_x = e^{-y} \cos x \text{ and } v_y = -(-e^{-y} \cos x) = e^{-y} \cos x \text{ so } u_x = v_y$$

$$u_y = -e^{-y} \sin x \text{ and } v_x = e^{-y} \sin x \text{ so } u_y = -v_x.$$

(R-equations satisfied on all of \mathbb{C} , partials continuous on \mathbb{C}
 $\Rightarrow f(z)$ is entire.

Note: $f(z) = -ie^{iz}$ which is clearly entire as the composition
of two entire functions.

$$(c) f(z) = \frac{2z+3}{z^2+8} = \frac{2z+3}{(z+2\sqrt{2}i)(z-2\sqrt{2}i)} \text{ is not diff. at } z = \pm 2\sqrt{2}i \\ \Rightarrow f \text{ is not an entire function.}$$

$$2. u(x,y) = xy + e^{-2y} \cos(2x)$$

$$u_x = y - 2e^{-2y} \sin(2x) \Rightarrow u_{xx} = -4e^{-2y} \cos(2x)$$

$$u_y = x - 2e^{-2y} \cos(2x) \Rightarrow u_{yy} = 4e^{-2y} \cos(2x)$$

$$\therefore u_{xx} + u_{yy} = 0 \quad \text{Partials continuous} \Rightarrow u \text{ is harmonic on } \mathbb{C}.$$

$$(\text{R-equations:}) \quad u_x = v_y \Rightarrow v_y = y - 2e^{-2y} \sin(2x)$$

$$\text{Int. w.r.t. } y \Rightarrow v(x,y) = \frac{1}{2}y^2 + e^{-2y} \sin(2x) + c(x)$$

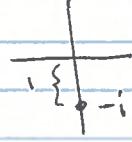
$$\text{Diff. w.r.t. } x \Rightarrow v_x = 2e^{-2y} \cos(2x) \stackrel{?}{=} -u_y = -x + 2e^{-2y} \cos(2x) \\ + c'(x)$$

$$\Rightarrow c'(x) = -x$$

$$\text{Int. w.r.t. } x \Rightarrow c(x) = -\frac{1}{2}x^2 + C$$

$$\therefore v(x,y) = \frac{1}{2}y^2 + e^{-2y} \sin(2x) - \frac{1}{2}x^2 + C \text{ is a harmonic conjugate} \\ \text{for any } C \in \mathbb{R}.$$

$$\begin{aligned}
 3. (a) (-i)^{1+2i} &= e^{(1+2i) \cdot \log(-i)} \quad \text{b/c } z^c = e^{c \log z} \\
 &= e^{(1+2i) \cdot (\ln| -i | + i \cdot -\frac{\pi}{2})} \\
 &= e^{(1+2i)(\ln 1 - i \frac{\pi}{2})} \\
 &= e^{(1+2i)(-i\pi)} \\
 &= e^{-i\pi} \cdot e^i \\
 &= -ie^i
 \end{aligned}$$



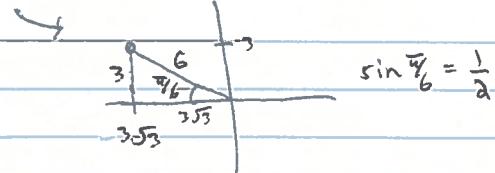
$$\begin{aligned}
 (b) \sin(\pi+i) &= \frac{e^{i(\pi+i)} - e^{-i(\pi+i)}}{2i} = \frac{1}{2i} (e^{i\pi} \cdot e^{-1} - e^{-i\pi} \cdot e^1) \\
 &= \frac{1}{2i} (-e^{-1} + e) = \frac{-i}{2} (e - \frac{1}{e}) \quad \text{since } \frac{1}{i} = -i
 \end{aligned}$$

or use $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

$$\therefore \sin(\pi+i) = \sin \pi \cos i + \cos \pi \cdot \sin i = -\sin i \quad \text{since } \sin \pi = 0.$$

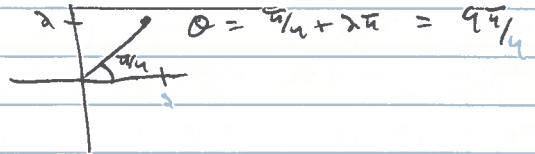
$$-\sin i = \frac{-1}{2i} (e^{i(i)} - e^{-i(i)}) = \frac{-1}{2i} (e^{-1} - e) = -\frac{i}{2} (e - \frac{1}{e}).$$

$$\begin{aligned}
 (c) \log(-3\sqrt{3} + 3i) &= \ln|-3\sqrt{3} + 3i| + i \operatorname{Arg}(-3\sqrt{3} + 3i) \\
 &= \ln 6 + i \frac{5\pi}{6}
 \end{aligned}$$



4. (a) $\log(2+2i) = \ln|2+2i| + i\theta$ where θ is the argument of $2+2i$ lying between $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

$$\begin{aligned}
 &= \ln \sqrt{8} + i \cdot \frac{9\pi}{4} \\
 &= \frac{1}{2} \ln 8 + i \cdot \frac{9\pi}{4}
 \end{aligned}$$



(b) False.

$$\begin{aligned}
 \log(i^2) &= \log(-1) = \ln|-1| + i \cdot 3\pi \quad \text{b/c } \frac{3\pi}{2} < 3\pi < \frac{7\pi}{2} \\
 &= 3\pi i
 \end{aligned}$$

but

$$\begin{aligned}
 2\log(i) &= 2(\ln|1| + i \cdot \frac{5\pi}{2}) \quad \text{b/c } \frac{3\pi}{2} < \frac{5\pi}{2} < \frac{7\pi}{2} \\
 &= 2 - i\frac{5\pi}{2} \\
 &= 5\pi i
 \end{aligned}$$

$$3\pi i \neq 5\pi i$$

$$5. (a) z = (1-t)1 + i(t), 0 \leq t \leq 1 \quad \text{or} \quad z = t(i-1) + 1, 0 \leq t \leq 1$$

$$\begin{aligned} \oint_C \bar{z} dz &= \int_0^1 (1-t-i) \cdot (i-1) dt = (i-1) \int_0^1 1-t-it dt \\ &= (i-1) \left[t - \frac{t^2}{2} - it^2 \Big|_0^1 \right] = (i-1) \left(1 - \frac{1}{2} - \frac{i}{2} \right) \\ &= (i-1) \left(\frac{1}{2} - \frac{i}{2} \right) = \boxed{i} \end{aligned}$$

$$(b) z = e^{-i\theta}, 0 \leq \theta \leq 2\pi$$

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{-i\theta}} \cdot -ie^{-i\theta} d\theta = \int_0^{2\pi} -i d\theta = \boxed{-2\pi i}$$

as expected.

$$(c) z = e^{i\theta}, 0 \leq \theta \leq 2\pi$$

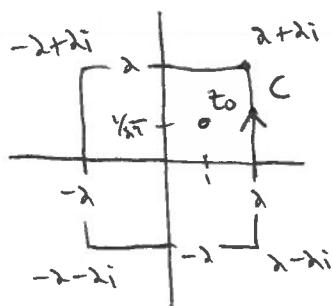
$$\begin{aligned} \oint_C \frac{1}{z^2} dz &= \int_0^{2\pi} \frac{1}{e^{2i\theta}} \cdot ie^{i\theta} d\theta = i \int_0^{2\pi} e^{-i\theta} d\theta = i \left[-\frac{1}{i} e^{-i\theta} \right]_0^{2\pi} \\ &= -(e^{-i2\pi} - e^0) \end{aligned}$$

$$(d) \int_{-i}^{i+2} ze^{z^2} dz = \frac{1}{2} e^{z^2} \Big|_{-i}^{i+2} \quad \text{by AD Thm.} \quad = \boxed{0}, \text{ as expected by AD Thm.}$$

$$\begin{aligned} &= \frac{1}{2} \left(e^{(i+2)^2} - e^{-i^2} \right) = \frac{1}{2} \left(e^{3+4i} - e^{-1} \right) = \frac{1}{2} \left(e^3 \underbrace{(e^{i\omega+4} + i\sin 4)}_{e^{ui}} - e^{-1} \right) \\ &= \boxed{\frac{1}{2} e^3 \cos 4 - \frac{1}{2} e^{-1} + i \cdot \frac{1}{2} e^3 \sin 4} \end{aligned}$$

$$6. (a)$$

$$f(z) = \frac{e^z}{z - (1 + \frac{1}{2}\pi i)} \quad \text{let } z_0 = 1 + \frac{1}{2}\pi i. \quad z_0 \text{ interior of } C.$$



use Cauchy-Integral formula with

$$g(z) = e^z \text{ and } z_0 = 1 + \frac{1}{2}\pi i$$

$$\oint_C \frac{g(z)}{z - z_0} dz = 2\pi i \cdot g(1 + \frac{1}{2}\pi i)$$

$$= 2\pi i \cdot e^{1 + \frac{1}{2}\pi i}$$

$$= 2\pi i \cdot e^1 \cdot e^{i\frac{1}{2}\pi} = 2\pi i \cdot e \cdot i = \boxed{-2\pi e}$$

6. (b) $f(z) = \frac{e^z}{z-(2+3i)}$ is analytic on and interior to C b/c
 $z = 2+3i$ lies exterior to C .

By quotient rule, f analytic on and inside C , so

$$\oint_C \frac{e^z}{z-(2+3i)} dz = 0 \text{ by the Cauchy-Goursat Thm.}$$

(c)

Let $g(z) = \frac{\cos z}{z^2+9}$. $g(z)$ is undefined at $z = \pm 3i$, but these lie outside of C . $\cos z$ is an entire function

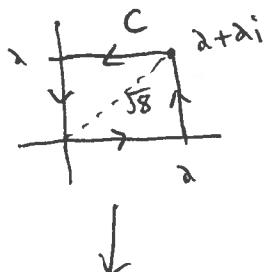
Thus, g is analytic on and inside C by the Quotient rule.

Using the Cauchy Integral formula, with $z_0 = -i$, (z_0 inside C)

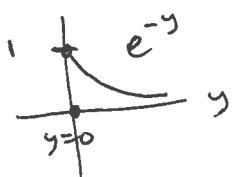
$$\begin{aligned} \oint_C f(z) dz &= \oint_C \frac{g(z)}{z+i} dz = 2\pi i \cdot g(-i) && \text{cos even} \\ &= 2\pi i \cdot \frac{\cos(-i)}{(-i)^2 + 9} = \frac{2\pi i}{8} \cos(i) \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} && \quad \quad \quad = \frac{\pi i}{4} \cdot \frac{e^{-i} + e^{i-i}}{2} = \boxed{\frac{\pi i}{8} (e + \frac{1}{e})} \end{aligned}$$

7. $\left| \oint_C e^{iz} - z^2 dz \right| \leq \oint_C |e^{iz} - z^2| dz \leq M \cdot L = 9 \cdot 8 = 72. \checkmark$

We use the ML Thm.:



$$L = 4 \cdot 2 = 8$$



Maximum value of e^{-y} occurs at $y=0$.

$$\begin{aligned} |e^{iz} - z^2| &\leq |e^{iz}| + |-z^2| \\ &= |e^{i(x+iy)}| + |z|^2 \\ &= |e^{ix} \cdot e^{-y}| + |z|^2 \\ &= e^{-y} + |z|^2 \\ &\leq e^0 + (\sqrt{8})^2 && \text{maximum modulus occurs at } z = 2+2i \\ &= 1 + 8 = 9. \quad \text{So } M = 9. \end{aligned}$$

8. $C = \text{unit circle}, z = e^{i\theta}, -\pi \leq \theta \leq \pi$

$$(a) \int_C \frac{e^{az}}{z} dz = 2\pi i \cdot e^0 \text{ by Cauchy Integral formula, using}$$

$$= 2\pi i \quad f(z) = e^{az} \text{ and } z_0 = 0.$$

✓ ↙ inside C
entire (chain rule)

$$(b) \int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{a(e^{i\theta})}}{e^{i\theta}} \cdot ie^{i\theta} d\theta = \int_{-\pi}^{\pi} ie^{a(\cos\theta + i\sin\theta)} d\theta$$

Euler's formula

$$= i \int_{-\pi}^{\pi} e^{a\cos\theta} \cdot e^{i\sin\theta} d\theta = i \int_{-\pi}^{\pi} e^{a\cos\theta} (\cos(a\sin\theta) + i\sin(a\sin\theta)) d\theta$$

$$= - \int_{-\pi}^{\pi} e^{a\cos\theta} \cdot \sin(a\sin\theta) d\theta + i \int_{-\pi}^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta) d\theta$$

$$= 2\pi i \text{ by part (a).}$$

\therefore First integral is 0 and second integral yields

$$2\pi i = \int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta$$

$$\stackrel{bc}{\therefore} \int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta = \int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta$$

$e^{a\cos\theta} \cdot \cos(a\sin\theta)$ is
an even function.

$$\therefore \int_0^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta) d\theta = \pi. \blacksquare$$

9. (a) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$ is false.

Choosing z_1 and z_2 in the 2nd and 3rd quadrants will lead to a contradiction.

For example, take $z_1 = -1+i$ and $z_2 = -1-i$. Then

$$\begin{aligned}\log\left(\frac{z_1}{z_2}\right) &= \log\left(\frac{-1+i}{-1-i}\right) = \log\left(\frac{(-1+i)(-1+i)}{(-1-i)(-1+i)}\right) = \log\left(\frac{-2i}{2}\right) \\ &= \log(-i) = \ln| -i | + i \cdot \arg(-i) = -\frac{\pi}{2}i\end{aligned}$$

but

$$\begin{aligned}\log z_1 - \log z_2 &= \log(-1+i) - \log(-1-i) \\ &= \ln\sqrt{2} + i\frac{3\pi}{4} - (\ln\sqrt{2} - i\frac{3\pi}{4}) = \frac{3\pi}{2}i.\end{aligned}$$

$$\text{So } -\frac{\pi}{2}i \neq \frac{3\pi}{2}i.$$

Another counter-example is $z_1 = -i$, $z_2 = i$ (from Alli M.), since $\log\left(\frac{-i}{i}\right) = \log(-1) = i\pi$ but

$$\log(-i) - \log(i) = -\frac{\pi}{2}i - \frac{\pi}{2}i = -i\pi.$$

(b) True. Since $e^{iz} = \cos z + i \sin z$ A+E C, we have

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z \text{ b/c } \cos(z) \text{ is even and } \sin(z) \text{ is odd.}$$

or

$$\begin{aligned}\cos z - i \sin z &= \frac{1}{2}[e^{iz} + e^{-iz}] - i \cdot \frac{1}{2i}[e^{iz} - e^{-iz}] \\ &= \cancel{\frac{1}{2}e^{iz}} + \frac{1}{2}e^{-iz} - \cancel{\frac{1}{2}e^{iz}} + \frac{1}{2}e^{-iz} \\ &= e^{-iz}. \quad \checkmark\end{aligned}$$

(c) True.

$$\begin{aligned}g(-z) &= e^{\cos(-z)} \cdot \sin(-z) \\&= e^{\cos z} \cdot -\sin z \\&= -e^{\cos z} \cdot \sin z \\&= -g(z).\end{aligned}$$

(d) True.

Recall that $z^c = e^{c \log z}$. So

$$\begin{aligned}z^{c_1} \cdot z^{c_2} &= e^{c_1 \log z} \cdot e^{c_2 \log z} = e^{c_1 \log z + c_2 \log z} \\&= e^{(c_1+c_2) \log z} = z^{c_1+c_2} \text{ by def.}\end{aligned}$$

(e) True. Use the AD Thm.

$f(z) = \frac{-1}{(z-1)^{2017}}$ is continuous on any domain not containing 1.

$F(z) = \frac{1}{2016} (z-1)^{-2016} = \frac{1}{2016(z-1)^{2016}}$ is an antiderivative for $f(z)$

since $F'(z) = f(z)$. F is analytic on any domain not containing $z=1$. $\therefore \int_C f(z) dz = 0$ by the AD Thm.

Since C does not pass through $z=1$.

Note that the Cauchy-Goursat Thm. does not apply b/c the contour might enclose $z=1$. The AD Thm. applies b/c we can

