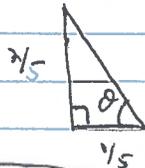
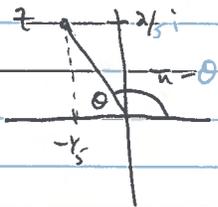


Solutions to Sample Questions for Exam 1

1. (a) $\frac{1+2i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{-5+10i}{25} = \frac{-1}{5} + \frac{2}{5}i$

Polar form: $r = \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}}$ or $\frac{\sqrt{5}}{5}$

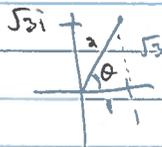


$\tan \theta = \frac{2/5}{1/5} = 2 \Rightarrow \theta = \tan^{-1}(2)$

$\therefore \text{Arg}(z) = \pi - \tan^{-1}(2)$

$\therefore z = \frac{1}{\sqrt{5}} e^{i(\pi - \tan^{-1}(2))}$

b) $(1 + \sqrt{3}i)^8$. First write $z = 1 + \sqrt{3}i$ as $z = \rho e^{i\theta}$



$\theta = \pi/3$
 $\sin \theta = \frac{\sqrt{3}}{2}$

Then $(1 + \sqrt{3}i)^8 = (\rho e^{i\theta})^8$

$= \rho^8 \cdot e^{8i\theta} = \rho^8 e^{i 2\pi/3}$ since $\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$

$= 2^8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$

$= 2^8 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$

$= 2^7 (-1 + i\sqrt{3}) = 128(-1 + i\sqrt{3})$

2. (a) $|z_1 z_2| = |z_1| \cdot |z_2|$

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

Then $z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$

$|z_1 z_2| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2}$

$= \sqrt{x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2}$

$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$= \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} = |z_1| \cdot |z_2| \quad \checkmark$

2. (b)

$$|z_1 z_2|^2 = z_1 z_2 \cdot \overline{z_1 z_2} = z_1 z_2 \cdot \overline{z_1} \cdot \overline{z_2} = \overline{z_1} \overline{z_2} \cdot z_1 z_2$$

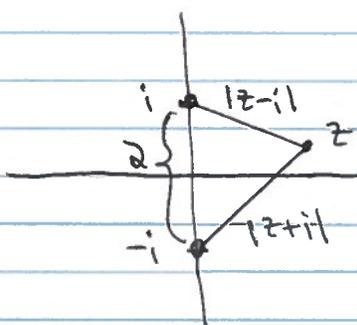
$$= |z_1|^2 \cdot |z_2|^2 = (|z_1| \cdot |z_2|)^2$$

$$\therefore |z_1 z_2|^2 = (|z_1| \cdot |z_2|)^2 \Rightarrow |z_1 z_2| = |z_1| \cdot |z_2|$$

b/c both quantities being squared are positive.

3. For this problem, use the fact that $|z_1 - z_2|$ equals the distance between z_1 and z_2 .

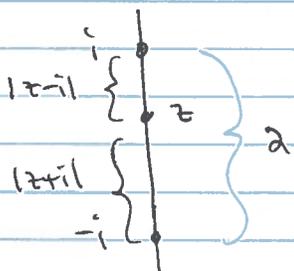
(a) Draw a Δ with vertices at i , $-i$, and z .



For any triangle with sides a, b , and c , we have $a \leq b+c$ and $b \leq a+c$ and $c \leq a+b$.

$$\therefore a \leq |z-i| + |z+i| \text{ as desired.}$$

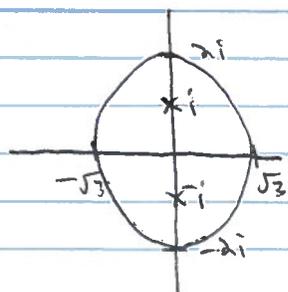
To have equality, we need the Δ to form a line, we also need z to be between $-i$ and i .



$$\therefore |z-i| + |z+i| = a \text{ iff } z = iy, -1 \leq y \leq 1.$$

(b) $|z+i| + |z-i| = 4$ describes an ellipse centered at the origin with semi-major axis of length 2 in the y -direction and "minor" " " $\sqrt{3}$ " " x -direction.

i and $-i$ are the focal points of the ellipse



The sum of the distance between z and i and between z and $-i$ is constant at 4. The set of points where the sum to two fixed points in the plane is constant is an ellipse (by definition).

Check that $z = 2i, -2i, \sqrt{3}, -\sqrt{3}$ all satisfy the equation.

3(b) cont.

It is possible to use $|z| = \sqrt{x^2 + y^2}$ to derive the equation of the ellipse.

$|z+i| + |z-i| = 4$ becomes $\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2} = 4$

Square both sides, simplify, square again, ... : $\frac{x^2}{3} + \frac{y^2}{4} = 1$

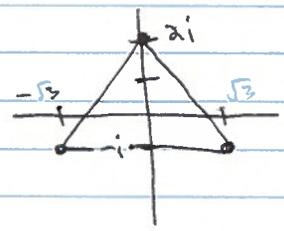
4. $z = -8i = 8e^{-i\pi/2}$ so $z^{1/3} = 8^{1/3} \cdot e^{i(-\pi/6 + \frac{2\pi}{3}k)}$, $k \in \{0, 1, 2\}$
 $= 2e^{i(-\pi/6 + \frac{2\pi}{3}k)}$

$k=0$: $2e^{-i\pi/6} = 2(\cos \pi/6 + i \sin \pi/6) = 2(\sqrt{3}/2 + i \cdot 1/2) = \sqrt{3} + i$
 $\sqrt{3} - i$ is the principal value of the root

$k=1$: $2e^{i\pi/2} = 2(\cos \pi/2 + i \sin \pi/2) = 2i$

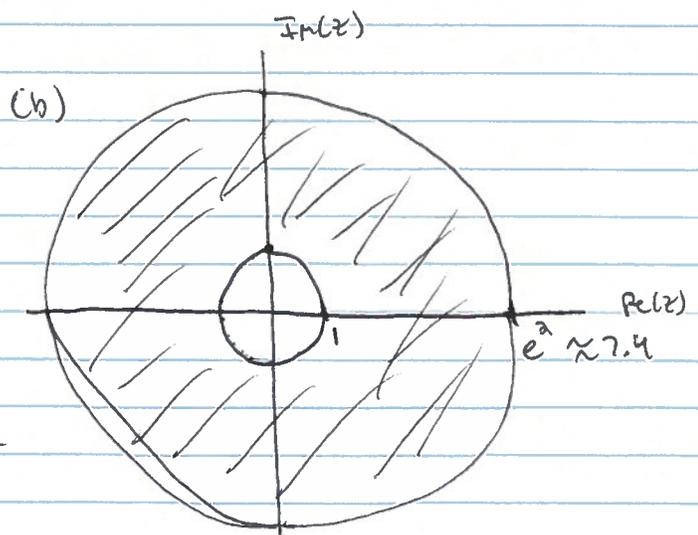
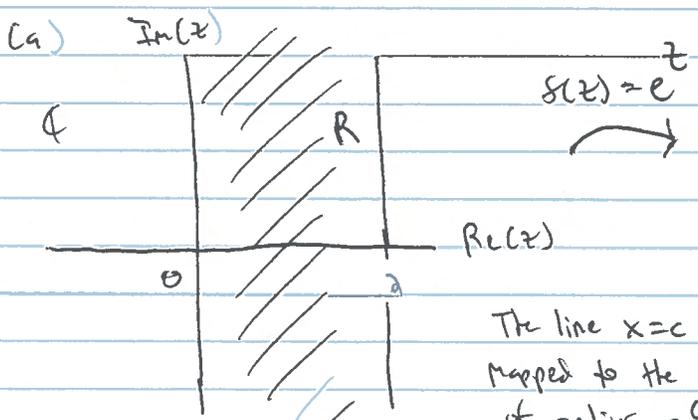
$k=2$: $2e^{i7\pi/6} = 2(\cos 7\pi/6 + i \sin 7\pi/6) = 2(-\sqrt{3}/2 - i \cdot 1/2) = -\sqrt{3} - i$

Roots: $\sqrt{3} - i, 2i, -\sqrt{3} - i$



Equilateral triangle on circle of radius 2.

5.



$w(z) = e^z$

The line $x=c$ is mapped to the circle of radius e^c .

$e^z = e^{x+iy} = e^x \cdot e^{iy}$
 so $|e^z| = |e^x| \cdot |e^{iy}| = e^x$

Annulus: $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$

6. Prove $\lim_{z \rightarrow 3i} \lambda z^2 - 4iz = -6$ with ϵ - δ definition.

Let $\epsilon > 0$ be given. Choose $\delta = \min \{1, \epsilon/10\}$.

$$\begin{aligned}
 \text{Consider } |f(z) - w| &= |\lambda z^2 - 4iz + 6| = \lambda |z^2 - 2iz + 3| \\
 &= \lambda |(z-3i)(z+i)| \\
 &= \lambda \underbrace{|z+i|}_{\text{bound this term}} \cdot \underbrace{|z-3i|}_{< \delta}
 \end{aligned}$$

Claim: $|z-3i| < 1 \Rightarrow |z| < 4$ and thus $|z+i| \leq |z|+1$
 $= |z|+1 < 5$.

Using $|z_1 + z_2| \geq ||z_1| - |z_2||$,
 we have

$$|z+3i| \geq ||z| - |-3i|| = ||z| - 3|$$

If $|z| \leq 3$, then $|z| < 4$ is clear.

If $|z| > 3$, then $|z|-3 > 0 \Rightarrow ||z|-3| = |z|-3$ (def. abs. value)

$$\begin{aligned}
 \therefore 1 > |z+3i| \geq |z|-3 \quad \text{or} \quad 1 > |z|-3 \Rightarrow |z| < 4. \\
 \Rightarrow |z+i| < 5.
 \end{aligned}$$

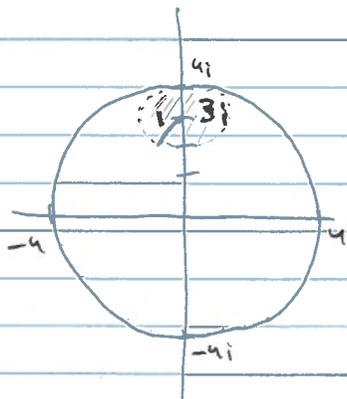
Punchline: Take $\delta = \min \{1, \epsilon/10\}$.

$$|f(z) - w| = \lambda |z+i| \cdot |z-3i| < \lambda \cdot 5 \cdot |z-3i|$$

$$< 10\delta$$

$$\leq 10 \cdot \frac{\epsilon}{10} = \epsilon \quad \text{whenever } 0 < |z-3i| < \delta.$$

Picture:



If $|z-3i| < 1$, $|z| < 4$,

↓
inside disk

↓
outer disk

$$f(z) = \frac{1}{z}$$

7.

$$(a) f(x+iy) = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}$$

$$\text{so } u(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{-y}{x^2+y^2}$$

$$\text{Polar form: } z = re^{i\theta} \Rightarrow \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r} (\cos(-\theta) + i \sin(-\theta)) \\ = \frac{1}{r} \cos\theta - i \frac{1}{r} \sin\theta$$

$$\text{so } u(r,\theta) = \frac{1}{r} \cos\theta \quad \text{and} \quad v(r,\theta) = -\frac{1}{r} \sin\theta$$

$$(b) \quad u_x = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad v_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$\therefore u_x = v_y \quad \checkmark \quad = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}, \quad v_x = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore u_y = -v_x \quad \checkmark$$

$$\text{Polar: } u_r = -\frac{1}{r^2} \cos\theta, \quad v_\theta = -\frac{1}{r} \cos\theta \Rightarrow r u_r = v_\theta \quad \checkmark$$

$$u_\theta = -\frac{1}{r} \sin\theta, \quad v_r = \frac{1}{r^2} \sin\theta \Rightarrow r v_r = -u_\theta \quad \checkmark$$

(c) Polar form is easiest:

$$\text{SCD THM. applicer } \Rightarrow f'(z) = e^{-i\theta} (u_r + i v_r)$$

on $\mathbb{C} - \{0\}$

$$= e^{-i\theta} \left(-\frac{1}{r^2} \cos\theta + i \frac{1}{r^2} \sin\theta \right)$$

$$= -\frac{1}{r^2} e^{-i\theta} (\cos\theta - i \sin\theta)$$

$$= -\frac{1}{r^2} e^{-i\theta} \cdot e^{-i\theta}$$

$$= -\frac{1}{r^2} e^{-i2\theta}$$

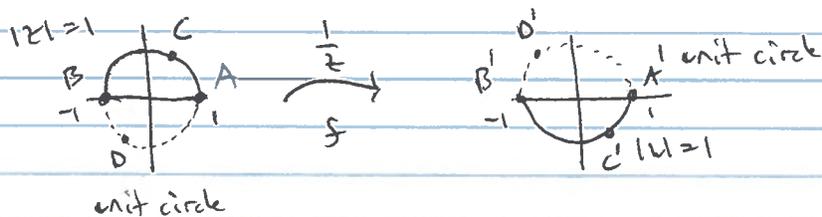
$$= \frac{-1}{r^2 e^{i2\theta}} = \frac{-1}{(re^{i\theta})^2} = \frac{-1}{z^2} \quad \checkmark$$

7. (d) $f(re^{i\theta}) = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} \Rightarrow r \mapsto \frac{1}{r}$ and $\theta \mapsto -\theta$
 invert modulus and negate argument

$\therefore |z|=1$ is mapped to unit circle

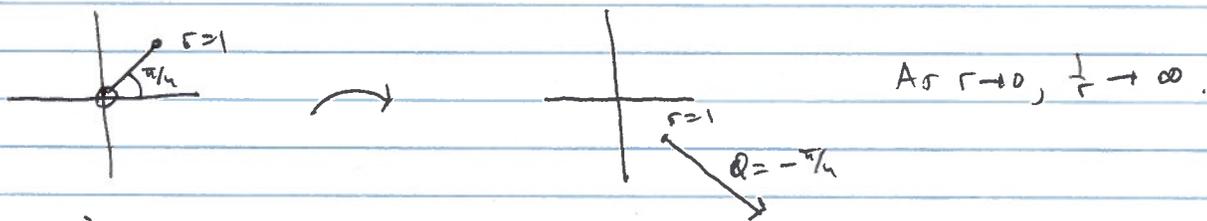
b/c $|f(z)| = \left| \frac{1}{z} \right| = \frac{1}{|z|} = \frac{1}{1} = 1$

and $\arg(f(z)) = -\arg(z)$



(e) $|z|=r$ is mapped to the circle $|w|=1/r$ b/c $|f(z)| = \frac{1}{|z|} = \frac{1}{r}$

(f) Image of $\theta = \pi/4, 0 < r \leq 1$ is ray $\theta = -\pi/4, 1 \leq \rho < \infty$



8. (a) $\lim_{z \rightarrow \infty} \frac{z^2}{(i+1)^2} = \frac{1}{i^2} = -1$

ps// $\lim_{z \rightarrow \infty} f(z) = w_0$ iff $\lim_{z \rightarrow 0} f(\frac{1}{z}) = w_0$ LIMI part 2

$f(z) = \frac{z^2}{(i+1)^2} \Rightarrow f(\frac{1}{z}) = \frac{(1/z)^2}{(1/z + 1)^2} = \frac{1/z^2}{(1/z + 1)^2} \cdot \frac{z^2}{z^2} = \frac{1}{(i+z)^2}$

so $\lim_{z \rightarrow 0} f(\frac{1}{z}) = \lim_{z \rightarrow 0} \frac{1}{(i+z)^2} = \frac{1}{i^2} = -1$. ✓

(b) $\lim_{z \rightarrow 2i} \frac{z}{z+4} = \infty$ iff $\lim_{z \rightarrow 2i} \frac{z+4}{z} = 0$ by LIMI part 1

iff $\lim_{z \rightarrow 2i} z + \frac{4}{z} = 0$ but $\lim_{z \rightarrow 2i} z + \frac{4}{z} = 2i + \frac{4}{2i} = 2i + \frac{4(-2i)}{4} = 0$. ✓

9. (a) $f(z) = y^2 + ix^2$ so $u(x,y) = y^2$ and $v(x,y) = x^2$

$u_x = 0$ and $v_y = 0$ so $u_x = v_y \quad \forall z \in \mathbb{C}$. ✓

$u_y = 2y$ and $v_x = 2x$ so $u_y = -v_x \Rightarrow 2y = -2x$ or $y = -x$

Since partials exist and are continuous on all of \mathbb{R}^2 , the SCD Thm. implies that $f'(z)$ exists for all (x,y) s.t. $y = -x$ or $\text{Im}(z) = -\text{Re}(z)$.

$f'(x-ix) = u_x + iv_x = 0 + i \cdot 2x = 2xi$

(b) $f(z) = x - 2xy + i(x^2 + y - y^2)$ so $u(x,y) = x - 2xy$, $v(x,y) = x^2 + y - y^2$

$u_x = 1 - 2y$ and $v_y = 1 - 2y \Rightarrow u_x = v_y \quad \forall z \in \mathbb{C}$

$u_y = -2x$ and $v_x = 2x \Rightarrow u_y = -v_x \quad \forall z \in \mathbb{C}$.

Since partials exist and are continuous on all of \mathbb{R}^2 , the SCD Thm. implies that $f'(z)$ exists for all $(x,y) \in \mathbb{R}^2$ or $\forall z \in \mathbb{C}$.

$f'(z) = u_x + iv_x = 1 - 2y + i(2x)$

Note: $f(z) = iz^2 + z$ so $f'(z) = 2iz + 1 = 2i(x+iy) + 1 = 1 - 2y + i(2x)$ ✓

10. $f(z) = \begin{cases} \frac{z^3}{(\bar{z})^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

(a) f is continuous at $z=0$ iff $\lim_{z \rightarrow 0} f(z) = f(0)$.

So we must show that $\lim_{z \rightarrow 0} \frac{z^3}{(\bar{z})^2} = f(0) = 0$.

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$.

$|f(z) - 0| = \left| \frac{z^3}{(\bar{z})^2} - 0 \right| = \frac{|z^3|}{|(\bar{z})^2|} = \frac{|z|^3}{|\bar{z}|^2} = \frac{|z|^3}{|z|^2} = |z|$

10 (a) cont. Set $\delta = \epsilon$.

Then, $|f(z) - 0| = |z| < \delta = \epsilon$ whenever $|z| < \delta$.

Note that $|f(z) - 0| < \epsilon$ holds trivially if $z=0$ b/c $f(0)=0$.

(b) $f'(0)$ does not exist.

$$\begin{aligned}
 f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - 0}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^3 / (h)^2}{h} = \lim_{h \rightarrow 0} \frac{h^2}{(h)^2}
 \end{aligned}$$

Case 1: Take $h = h_1, h_2 = 0$ (real axis)

The the limit becomes $\lim_{h_1 \rightarrow 0} \frac{h_1^2}{h_1^2} = 1$

Case 2: Take $h = h_1 + ih_1, h_2 = h_1$ (diagonal)

The the limit becomes $\lim_{h_1 \rightarrow 0} \frac{(h_1 + ih_1)^2}{(h_1 - ih_1)^2} = \lim_{h_1 \rightarrow 0} \frac{h_1^2 (1+i)^2}{h_1^2 (1-i)^2} = \frac{2i}{-2i} = -1$

Since the limits from different directions are not equal, the limit does not exist.

$\therefore f'(0)$ D.N.E.