MATH 305 Complex Analysis

Exam #2 SOLUTIONS April 21, 2016 Prof. G. Roberts

1. Consider the functions $f(z)$ and $g(z)$ given below. One of these functions is entire, while the other is nowhere analytic. Determine which is which. Be sure to **show** that each function has the given property. (12 pts.)

$$f(z) = e^{2y}\cos(2x) + ie^{2y}\sin(2x), \qquad g(z) = e^{-2y}\cos(2x) + ie^{-2y}\sin(2x),$$

Answer: f is nowhere analytic and g is entire.

We check the Cauchy-Riemann equations and apply the SCD Theorem. For $f(z)$, we have $u = e^{2y}\cos(2x)$ and $v = e^{2y}\sin(2x)$, so that

$$u_x = -2e^{2y}\sin(2x) \quad \text{and} \quad v_y = 2e^{2y}\sin(2x).$$

It follows that $u_x = v_y$ only if $\sin(2x) = 0$ or $x = (n\pi)/2, n \in \mathbb{Z}$. Similarly, we have

$$u_y = 2e^{2y}\cos(2x) \quad \text{and} \quad v_x = 2e^{2y}\cos(2x),$$

so that $u_y = -v_x$ is true only if $\cos(2x) = 0$ or $x = \pi/4 + (n\pi)/2, n \in \mathbb{Z}$. Since $\cos^2(2x) + \sin^2(2x) = 1 \neq 0$, it is clear that the Cauchy-Riemann equations can not *both* be satisfied simultaneously. Thus, $f'(z)$ does not exist for any $z \in \mathbb{C}$, which means f is nowhere analytic.

For $g(z)$, we have $u = e^{-2y}\cos(2x)$ and $v = e^{-2y}\sin(2x)$, so that

$$u_x = -2e^{-2y}\sin(2x) = v_y \quad \text{and} \quad u_y = -2e^{-2y}\cos(2x) = -v_x,$$

so that the Cauchy-Riemann equations are satisfied on all of $\mathbb{C}$. Since the partial derivatives are continuous on the entire plane, the SCD theorem then implies that $f'(z)$ exists for all $z \in \mathbb{C}$. Thus, g is an entire function. (Note that $g(z) = e^{2iz}$, which is the composition of the entire functions $2iz$ and e^z.)

2. Show that $u(x, y) = -y - x^3 + 3xy^2$ is a harmonic function and find a harmonic conjugate $v(x, y)$. (12 pts.)

Answer: First, we check that u satisfies Laplace's equation $u_{xx} + u_{yy} = 0$. We have $u_x = -3x^2 + 3y^2$, so that $u_{xx} = -6x$. Likewise, we have $u_y = -1 + 6xy$, so that $u_{yy} = 6x$. It follows that $u_{xx} + u_{yy} = 0$. Since the first and second partial derivatives of u exist and are continuous, u is a harmonic function.

To find a harmonic conjugate v, we begin with the first Cauchy-Riemann equation $u_x = v_y$. This implies that $v_y = -3x^2 + 3y^2$. Integrating this equation with respect to y gives

$$v(x, y) = -3x^2y + y^3 + c(x), \tag{1}$$

where $c(x)$ is some unknown function of the single variable x. If we now differentiate v with respect to x, we find $v_x = -6xy + c'(x)$. By the second Cauchy Riemann equation, $u_y = -v_x$, we must have

$$-1 + 6xy = 6xy - c'(x) \quad \text{or} \quad c'(x) = 1.$$

Integrating this last equation with respect to x gives $c(x) = x + c$, where $c \in \mathbb{R}$ is an arbitrary constant. Returning to our v in equation (1), we obtain $v(x, y) = -3x^2y + y^3 + x + c$.

3. Answer the following questions. Be sure to show your work. (13 pts.)

(a) Compute the principal value of $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{3+i}$.

Answer: $-e^{\pi/3}$. Use the formula $z^c = e^{c\,\mathrm{Log}\,z}$. We first compute that $|1/2 - i\sqrt{3}/2| = 1$ (lies on the unit circle) and $\mathrm{Arg}(1/2 - i\sqrt{3}/2) = -\pi/3$. It follows that $\mathrm{Log}(1/2 - i\sqrt{3}/2) = \ln 1 - i\pi/3 = -i\pi/3$. Thus,

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{3+i} = e^{(3+i)\cdot(-i\pi/3)} = e^{-i\pi + \pi/3} = e^{-i\pi} \cdot e^{\pi/3} = -e^{\pi/3},$$

since $e^{-i\pi} = -1$.

(b) Let $z = re^{i\theta}$ with $r > 0$. Show that $e^{\log z} = z$ for **any** value of the multiple-valued function $\log z$.

Answer: We have

$$\begin{aligned} e^{\log z} &= e^{\ln r + i(\theta + 2\pi n)}, \text{ where } n \in \mathbb{Z} \\ &= e^{\ln r} \cdot e^{i\theta} \cdot e^{i2\pi n} \\ &= re^{i\theta} \quad \text{since } e^{i2\pi n} = 1 \text{ for } n \in \mathbb{Z} \\ &= z. \quad \text{QED} \end{aligned}$$

Note that $r > 0$ is important to assume since $\log 0$ is undefined. Also, $e^{i2\pi n} = \cos(2\pi n) + i\sin(2\pi n) = 1$, since n is an integer.

4. Let C denote the circle of radius four centered at the origin, traversed in the counterclockwise direction. Evaluate the following contour integral in **TWO** different ways. For method one, use a parametrization of C and evaluate the contour integral directly. For method two, use the Cauchy integral formula. (14 pts.)

$$\oint_C \frac{z^2 + 3}{z}\, dz$$

Answer: $6\pi i$.

First, let $z = 4e^{i\theta}, 0 \le \theta \le 2\pi$ parametrize the circle. Evaluating the contour integral directly gives

$$\begin{aligned} \oint_C \frac{z^2+3}{z}\, dz &= \int_0^{2\pi} \frac{16e^{2i\theta} + 3}{4e^{i\theta}} \cdot 4ie^{i\theta}\, d\theta \\ &= i\int_0^{2\pi} 16e^{2i\theta} + 3\, d\theta \\ &= i\left(\frac{8}{i}e^{2i\theta} + 3\theta\Big|_0^{2\pi}\right) \\ &= 8e^{2i\theta} + 3i\theta\Big|_0^{2\pi} \\ &= 8e^{4\pi i} + 6\pi i - 8 \\ &= 6\pi i. \end{aligned}$$

Second, we apply the Cauchy integral formula $f(z_0) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z - z_0}\,dz$ with $f(z) = z^2 + 3$ (an entire function) and $z_0 = 0$. This yields

$$\oint_C \frac{z^2+3}{z}\,dz \;=\; 2\pi i \cdot f(0) \;=\; 2\pi i \cdot 3 \;=\; 6\pi i,$$

as expected.

5. Compute each of the following integrals. Be sure to specify what theorem or formula you are using. (21 pts.)

   (a) $\displaystyle\int_C \overline{z}^2\,dz$, where C is the line segment from i to $-i$.

   Answer: $2i/3$.
   Since $\overline{z}^2$ is not analytic, nor does it have an antiderivative, we must compute the integral directly. To parametrize the line segment, we let $z = (1-t)i + t(-i) = i - 2it = i(1-2t), 0 \le t \le 1$. Then,

$$\begin{aligned}
\oint_C \overline{z}^2\,dz &= \int_0^1 [i(2t-1)]^2 \cdot (-2i)\,dt \\
&= -2i\int_0^1 i^2(2t-1)^2\,dt \\
&= 2i\int_0^1 (2t-1)^2\,dt \\
&= 2i\,\frac{(2t-1)^3}{3}\cdot\frac{1}{2}\Big|_0^1 \qquad (u\text{-sub with } u = 2t-1) \\
&= \frac{i}{3}\,(2t-1)^3\Big|_0^1 \\
&= \frac{i}{3}\left(1^3 - (-1)^3\right) \\
&= \frac{2i}{3}.
\end{aligned}$$

(b) $\displaystyle\int_{i\pi}^{\pi} \cos(iz)\, dz.$

Answer: $\sinh(\pi)$.

We use the AD Theorem.

$$\begin{aligned}
\int_{i\pi}^{\pi} \cos(iz)\, dz &= \frac{1}{i}\sin(iz)\Big|_{i\pi}^{\pi} \\
&= \frac{1}{i}\left(\sin(i\pi) - \sin(-\pi)\right) \\
&= \frac{1}{i}\sin(i\pi) \\
&= \frac{1}{i}\cdot\frac{1}{2i}\left(e^{i(i\pi)} - e^{-i(i\pi)}\right) \\
&= -\frac{1}{2}\left(e^{-\pi} - e^{\pi}\right) \\
&= \frac{1}{2}\left(e^{\pi} - e^{-\pi}\right) \\
&= \sinh(\pi).
\end{aligned}$$

(c) $\displaystyle\oint_C \frac{z^2+1}{e^z}\, dz$, where C is the square with vertices $2+2i$, $-2+2i$, $-2-2i$ and $2-2i$, traversed in the counterclockwise direction.

Answer: 0. Since both z^2+1 and e^z are entire functions, *and* since $e^z \neq 0$ for any $z \in \mathbb{C}$, the quotient rule implies that $(z^2+1)/e^z$ is an entire function. By the Cauchy-Goursat Theorem, the contour integral is 0. To see why $e^z \neq 0$ for any $z \in \mathbb{C}$, recall that $|e^z| = e^x > 0$ for any $x \in \mathbb{R}$. Since the modulus is always positive, the value of the function e^z can never vanish.

6. Without computing the integral, show that

$$\left|\oint_C (\overline{z}e^z - i)\, dz\right| \leq 6(1+\sqrt{2}\,e),$$

where C is the rectangle with vertices -1, 1, $1+i$ and $-1+i$, traversed in the counterclockwise direction. (12 pts.)

Answer: We use the ML-theorem. The length L of the contour is equal to the perimeter of the rectangle, which is $2+1+2+1=6$. To bound the modulus of the integrand, we have

$$\begin{aligned}
|\overline{z}e^z - i| &\leq |\overline{z}e^z| + |-i| \quad \text{(triangle inequality)} \\
&= |\overline{z}|\cdot|e^z| + 1 \\
&= |z|\cdot e^x + 1 \\
&\leq \sqrt{2}\cdot e^x + 1 \quad \text{(since } 1+i \text{ and } -1+i \text{ are the furthest points from the origin)} \\
&\leq \sqrt{2}\cdot e^1 + 1 \quad \text{(since } x \leq 1 \text{ on } C\text{)} \\
&= 1+\sqrt{2}\,e.
\end{aligned}$$

Consequently,

$$\left|\oint_C (\overline{z}e^z - i)\, dz\right| \leq M\cdot L = 6(1+\sqrt{2}\,e). \quad \text{QED}$$

7. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample**. (16 pts.)

(a) $\cos^2 z + \sin^2 z = 1$ for any $z \in \mathbb{C}$.

Answer: TRUE.

Using the definitions of $\cos z$ and $\sin z$, we have

$$\begin{aligned}
\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\
&= \frac{1}{4}\left(e^{2iz}+2+e^{-2iz}\right) + \frac{1}{-4}\left(e^{2iz}-2+e^{-2iz}\right) \\
&= \frac{1}{4}\left(e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}\right) \\
&= \frac{1}{4}\cdot 4 \\
&= 1. \quad \text{QED}
\end{aligned}$$

(b) If the branch $\log z = \ln r + i\,\theta$ $(r > 0, -\frac{\pi}{4} < \theta < \frac{7\pi}{4})$ is specified for the logarithmic function, then $\log(z^2) = 2\log z$ for any z in the domain of both functions.

Answer: FALSE.

The equation $\log(z^2) = 2\log z$ will not hold for any z with $7\pi/8 < \arg z < 7\pi/4$. For example, if $z = -1$, then we have

$$\log(z^2) = \log(1) = \ln 1 + i\cdot 0 = 0 \quad (\text{since } 0 \in (-\pi/4, 7\pi/4)),$$

while

$$2\log z = 2\log(-1) = 2(\ln 1 + i\pi) = 2\pi i, \quad (\text{since } \pi \in (-\pi/4, 7\pi/4)).$$

Since $0 \neq 2\pi i$, we have a counterexample.

Similarly, if $z = -i$, then

$$\log(z^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad (\text{since } \pi \in (-\pi/4, 7\pi/4)),$$

while

$$2\log z = 2\log(-i) = 2(\ln 1 + i\,3\pi/2) = 3\pi i, \quad (\text{since } 3\pi/2 \in (-\pi/4, 7\pi/4)).$$

Since $\pi i \neq 3\pi i$, we have another counterexample.

Because $2\log z$ will double the imaginary part (the argument of z in the specified branch), any z with $7\pi/8 < \arg z < 7\pi/4$ will result in an imaginary part for $2\log z$ *outside* the specified region. For such a z, $\log(z^2)$ will always have an imaginary part that is 2π less than that of $2\log z$, as is the case with the two counterexamples above.