MATH 305 Complex Analysis

Exam #1 SOLUTIONS **March 3, 2016** **Prof. G. Roberts**

1. [13 pts.] Circle **all** of the following complex numbers that lie on the unit circle. There may be anywhere from zero to six correct choices.

   (a) $-i$

   (b) $\dfrac{4}{5} - \dfrac{3}{5}i$

   (c) $\dfrac{2}{3-i}$

   (d) $\sqrt{2}\, e^{i\theta}$ for any $\theta \in \mathbb{R}$

   (e) $e^{i\sqrt{2}\,\theta}$ for any $\theta \in \mathbb{R}$

   (f) $\dfrac{\overline{z}}{z}$ for any $z \in \mathbb{C} - \{0\}$

   Answer: Choices **(a)**, **(b)**, **(e)**, and **(f)** all lie on the unit circle since they have a modulus of one. For **(b)**, we have

$$\left| \frac{4}{5} - \frac{3}{5}i \right| \;=\; \sqrt{(4/5)^2 + (-3/5)^2} \;=\; \sqrt{16/25 + 9/25} \;=\; 1,$$

   while for **(e)**, we have

$$|e^{i\sqrt{2}\,\theta}| \;=\; |\cos(\sqrt{2}\,\theta) + i\sin(\sqrt{2}\,\theta)| \;=\; \sqrt{\cos^2(\sqrt{2}\,\theta) + \sin^2(\sqrt{2}\,\theta)} \;=\; 1.$$

   For choice **(f)**, note that

$$\left| \frac{\bar{z}}{z} \right| \;=\; \frac{|\bar{z}\,|}{|z|} \;=\; \frac{|z|}{|z|} \;=\; 1.$$

   Choice **(c)** has a modulus of $\sqrt{10}/5$ and choice **(d)** has a modulus of $\sqrt{2}$.

2. Let z be the complex number $z = -2 + 2\sqrt{3}\,i$.

a) Write z in polar form, $z = re^{i\theta}$. (4 pts.)

Answer: We have $r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$. By drawing a 30-60-90 triangle in the second quadrant, we find that $\mathrm{Arg}(z) = \pi - \pi/3 = 2\pi/3$. Therefore, $z = 4e^{i\frac{2\pi}{3}}$.

b) Using your answer to part **a)**, find all of the roots $z^{1/4}$ in rectangular coordinates $x + iy$. Draw a sketch of the roots in the complex plane, including the polygon whose vertices are located at the roots. (10 pts.)

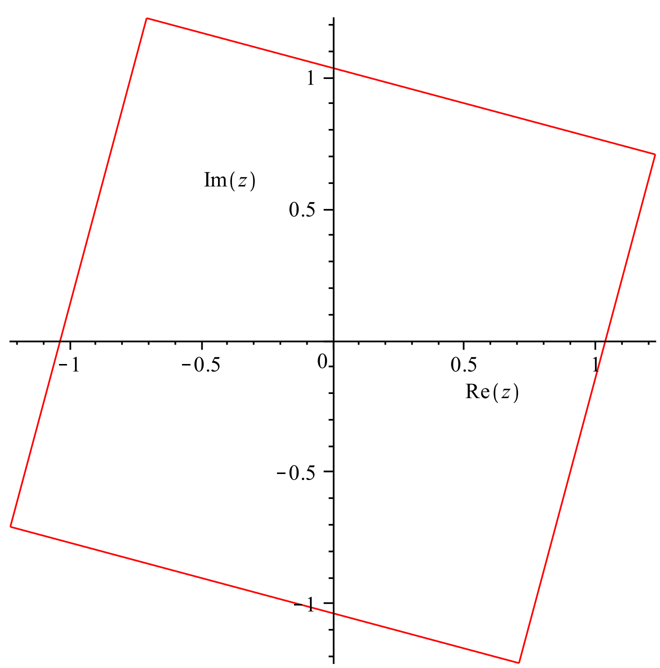
Answer: The roots are

$$\pm\sqrt{2}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) \quad \text{and} \quad \pm\sqrt{2}\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right).$$

Using the polar form of z, we find that

$$\begin{aligned}
\left(4e^{i\frac{2\pi}{3}}\right)^{1/4} &= 4^{1/4}\cdot e^{i\left(\frac{2\pi}{12}+\frac{2\pi k}{4}\right)}, \quad k = 0,1,2,3 \\
&= \sqrt{2}\,e^{i\left(\frac{\pi}{6}+\frac{\pi k}{2}\right)}, \quad k = 0,1,2,3 \\
&= \sqrt{2}\,e^{i\frac{\pi}{6}}, \sqrt{2}\,e^{i\frac{2\pi}{3}}, \sqrt{2}\,e^{i\frac{7\pi}{6}}, \sqrt{2}\,e^{i\frac{5\pi}{3}} \\
&= \sqrt{2}\,e^{i\frac{\pi}{6}}, \sqrt{2}\,e^{i\frac{2\pi}{3}}, -\sqrt{2}\,e^{i\frac{\pi}{6}}, -\sqrt{2}\,e^{i\frac{2\pi}{3}}
\end{aligned}$$

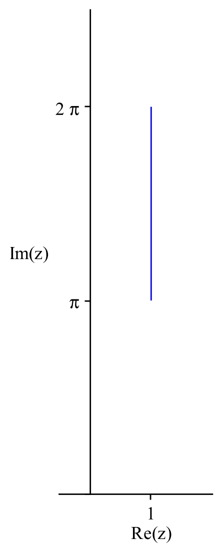
using the fact that $e^{i\pi} = -1$. Converting these polar expressions into rectangular coordinates using Euler's formula gives the answer above. The four roots form a square in the complex plane that is rotated 30° ccw from the positive x-axis and lies on the circle of radius $\sqrt{2}$ centered at the origin.

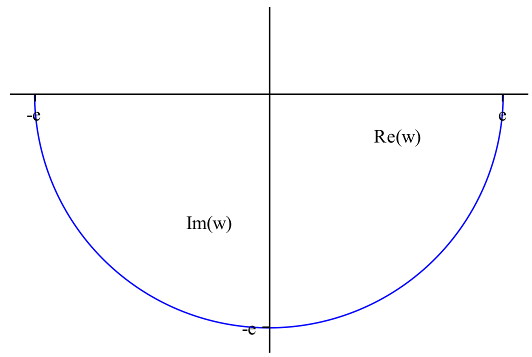


3. [13 pts.] **Sets in $\mathbb{C}$**

(a) Let R be the set $\{z \in \mathbb{C} : \mathrm{Re}(z) = 1, \pi \leq \mathrm{Im}(z) \leq 2\pi\}$. Sketch the set R in the complex plane. Then sketch and describe the image of R under the map $f(z) = e^z$. Be sure to label your graphs and axes carefully.

Answer:



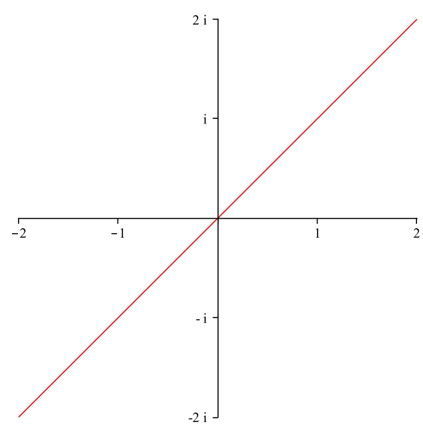


The set R is simply a line segment with real part equal to 1 and imaginary part between π and 2π. Using the fact that $e^z = e^x \cdot e^{iy}$, we see that the image of R under e^z is a semi-circle of radius $e^1 = e$ with angles ranging between π and 2π as shown.

(b) Describe in words and sketch the set of points $z \in \mathbb{C}$ satisfying the equation

$$|z - i| \;=\; |z - 1|.$$

Answer:



Recall that $|z_1 - z_2|$ represents the distance between the two complex numbers z_1 and z_2. Thus, the set of $z \in \mathbb{C}$ satisfying $|z - i| = |z - 1|$ is the set of all complex numbers that are equidistant from 1 and i in the complex plane. This is the perpendicular bisector of the line segment between 1 and i, or the set of points whose real and imaginary parts are equal:

$$\{z \in \mathbb{C} : \mathrm{Re}(z) = \mathrm{Im}(z)\} \quad \text{or} \quad \{z = x + iy : y = x\}$$

Alternatively, we have $|z-i| = |x+i(y-1)| = \sqrt{x^2+(y-1)^2}$ and $|z-1| = |(x-1)+iy| = \sqrt{(x-1)^2+y^2}$. Solving $\sqrt{x^2+(y-1)^2} = \sqrt{(x-1)^2+y^2}$ yields

$$x^2+(y-1)^2 \;=\; (x-1)^2+y^2 \quad\Longrightarrow\quad x^2+y^2-2y+1 \;=\; x^2-2x+1+y^2,$$

which simplifies to $y = x$.

4. [12 pts.] Use the ϵ-δ definition of the limit to prove rigorously that $\lim_{z\to 4i} 3i\,\overline{z} - 7 = 5$.

Answer: Let $\epsilon > 0$ be given. If $f(z) = 3i\,\overline{z} - 7$ and $w = 5$, we want to find a $\delta = \delta(\epsilon)$ so that $|f(z) - w| < \epsilon$ whenever $0 < |z - 4i| < \delta$. Using properties of modulus and conjugate, we have

$$\begin{aligned}
|f(z) - w| &= |3i\,\overline{z} - 7 - 5| \\
&= |3i\,\overline{z} - 12| \\
&= |3i(\overline{z} + 4i)| \\
&= |3i| \cdot |\overline{z} + 4i| \\
&= 3 \cdot |\overline{z + 4i}| \\
&= 3|z - 4i|.
\end{aligned}$$

Choose $\delta = \epsilon/3$. Then,

$$|f(z) - w| \;=\; 3|z-4i| \;<\; 3\delta \;=\; 3\cdot\frac{\epsilon}{3} \;=\; \epsilon$$

whenever $0 < |z - 4i| < \delta$. QED

5. [14 pts.] Evaluate the following limits, if they exist. Show your work, making sure to justify your answers thoroughly.

(a) $\displaystyle\lim_{z\to\infty} \frac{6i(z^2+1)}{(3z-i)^2}$

Answer: $2i/3$. Using the LIPI theorem, and later the BLT, we have

$$\begin{aligned}
\lim_{z\to\infty} \frac{6i(z^2+1)}{(3z-i)^2} &= \lim_{z\to\infty} \frac{6iz^2+6i}{9z^2-6iz-1} \\
&= \lim_{z\to 0} \frac{6i\left(\frac{1}{z}\right)^2 + 6i}{9\left(\frac{1}{z}\right)^2 - 6i\cdot\frac{1}{z} - 1} \\
&= \lim_{z\to 0} \frac{6i + 6iz^2}{9 - 6iz - z^2} \\
&= \frac{6i+0}{9-0-0} \;=\; \frac{6i}{9} \;=\; \frac{2i}{3}.
\end{aligned}$$

(b) $\displaystyle\lim_{z\to 0} \frac{\text{Im}(z)}{z}$

Answer: This limit does not exist, which can be seen by taking the limit in the real and imaginary directions. Suppose $z = x$ (so $y = 0$) and we take the limit along the real axis. Then,

$$\lim_{z\to 0} \frac{\text{Im}(z)}{z} = \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0.$$

On the other hand, if z is pure imaginary, $z = iy$ (so $x = 0$), we obtain

$$\lim_{z\to 0} \frac{\text{Im}(z)}{z} = \lim_{y\to 0} \frac{y}{iy} = \lim_{y\to 0} \frac{1}{i} = \frac{1}{i} = -i.$$

Since the two limits do not agree, the limit does not exist.

6. [18 pts.] For each of the functions below, find the points in the complex plane (if any) where $f'(z)$ exists and give a formula (simplified) for the derivative at those points.

(a) $f(z) = x^2 + 3y^2 + i(2xy + \cos(4x))$.

Answer: We have $u(x, y) = x^2 + 3y^2$ and $v(x, y) = 2xy + \cos(4x)$. Checking the Cauchy-Riemann equations, we have

$$u_x = 2x = v_y,$$

but $u_y = 6y$ and $v_x = 2y - 4\sin(4x)$. Solving the equation $u_y = -v_x$ yields

$$6y = -2y + 4\sin(4x) \quad \text{or} \quad y = \frac{1}{2}\sin(4x).$$

Since the partial derivatives exist and are continuous on the whole plane, the SCD theorem guarantees that $f'(z)$ exists for all complex numbers $z = x + iy$ satisfying $y = \frac{1}{2}\sin(4x)$.

To compute the value of the derivative at these points, we find that

$$f'(z) = u_x + iv_x = 2x + i(2y - 4\sin(4x)) = 2x - i3\sin(4x) \text{ or } 2x - i6y.$$

(b) $f(z) = (\overline{z})^2 + iz$

Answer: First we compute the real and imaginary parts of f by substituting in $z = x + iy$. This gives

$$f(z) = (x - iy)^2 + i(x + iy) = x^2 - y^2 - 2xyi + xi - y = x^2 - y^2 - y + i(x - 2xy),$$

so that $u = x^2 - y^2 - y$ and $v = x - 2xy$. Checking the Cauchy-Riemann equations, we have $u_x = 2x$ and $v_y = -2x$ so that $u_x = v_y$ gives

$$2x = -2x \implies x = 0.$$

Similarly, we have $u_y = -2y - 1$ and $v_x = 1 - 2y$ so that $u_y = -v_x$ implies

$$-2y - 1 = -1 + 2y \implies y = 0.$$

Thus, $f'(z)$ can only exist at the origin $(0, 0)$ or $z = 0$. By the SCD Theorem, since the partial derivatives are continuous at the origin, we know that $f'(0)$ exists and is found via

$$f'(0) = u_x(0, 0) + iv_x(0, 0) = 0 + i \cdot 1 = i.$$

7. [16 pts.] **Complex Analysis Potpourri:**

(a) Give an example of two complex numbers z_1, z_2 such that $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$.

Answer: The key is to take complex numbers whose real part is negative. For instance, if $z_1 = -1 + i$ and $z_2 = -1$, then

$$\text{Arg}(z_1 z_2) \;=\; \text{Arg}(1 - i) \;=\; -\frac{\pi}{4},$$

but

$$\text{Arg}(z_1) \;=\; \frac{3\pi}{4},\ \text{Arg}(z_2) \;=\; \pi \quad \text{implies} \quad \text{Arg}(z_1) + \text{Arg}(z_2) \;=\; \frac{7\pi}{4} \neq -\frac{\pi}{4}.$$

Other examples include $z_1 = -1$ and $z_2 = i$, and even simply $z_1 = z_2 = -1$.

(b) Which point on the Riemann sphere corresponds to ∞?

Answer: The North Pole.

(c) If the Cauchy-Riemann equations are satisfied for the function $f(z)$ at the point z_0, then $f'(z_0)$ exists.

Answer: This statement is FALSE. Just satisfying the Cauchy-Riemann equations alone is not enough to guarantee differentiability. The partial derivatives must also be continuous at z_0.

(d) If $f(z) = u(r, \theta) + iv(r, \theta)$, what are the Cauchy-Riemann equations in polar form?

Answer: $ru_r = v_\theta$ and $-rv_r = u_\theta$.

(e) Suppose that $u(x, y)$ is a function of two variables. State the limit definition of $u_y(1, 2)$.

Answer:

$$u_y(1, 2) \;=\; \lim_{h \to 0} \frac{u(1, 2 + h) - u(1, 2)}{h}.$$