

Section 22: Sufficient Conditions for Differentiability

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Class Notes for MATH 305

Complex Analysis

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Cauchy-Riemann Equations

Recall: The **Cauchy-Riemann Equations** must be satisfied in order for a complex function $f(z) = u(x, y) + i v(x, y)$ to be differentiable at a point.

$$u_x = v_y, \quad u_y = -v_x$$

They are derived by evaluating the limit definition of the derivative approaching in the real direction and the pure imaginary direction, and then equating the two results.

Let $z_0 = x_0 + i y_0$. If $f'(z_0)$ exists, then its value is $u_x + i v_x$, where each partial is evaluated at (x_0, y_0) .

Important: The Cauchy-Riemann equations are **necessary** conditions for $f'(z_0)$ to exist, but they are not sufficient.

Main Theorem in the Section

Theorem (SCD – Sufficient Conditions for Differentiability)

Suppose that $f(z) = u(x, y) + i v(x, y)$ is defined in a neighborhood of $z_0 = x_0 + i y_0$ and that

- 1 u_x, u_y, v_x, v_y exist everywhere in the neighborhood,
- 2 $u_x = v_y, u_y = -v_x$ at (x_0, y_0) , and
- 3 u_x, u_y, v_x, v_y are **continuous** at (x_0, y_0) .

Then $f'(z_0)$ exists and $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$.

Note: It is possible to satisfy the Cauchy-Riemann equations at a point, yet **not** be differentiable there (see Exercise #6 in Section 23 for such an example – HW). The point of the theorem is that continuity of the partial derivatives, not just satisfying the Cauchy-Riemann equations, is also required to insure that the derivative $f'(z_0)$ exists.

Proof of SCD Theorem

Proof: Assume that the hypotheses of the theorem are true. We will show that

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = u_x(x_0, y_0) + i v_x(x_0, y_0),$$

thereby proving that the derivative exists and the value is the one given in the theorem.

The proof relies on the multivariable version of [Taylor's Theorem](#). If both partial derivatives of a function $u(x, y)$ exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0) , then we can evaluate u nearby using the expression

$$\begin{aligned} u(x_0 + h_1, y_0 + h_2) &= u(x_0, y_0) + u_x(x_0, y_0)h_1 + u_y(x_0, y_0)h_2 \\ &\quad + \alpha_1(h_1, h_2)h_1 + \alpha_2(h_1, h_2)h_2 \end{aligned}$$

where α_1 and α_2 satisfy

$$\lim_{(h_1, h_2) \rightarrow (0,0)} \alpha_1(h_1, h_2) = 0 \quad \text{and} \quad \lim_{(h_1, h_2) \rightarrow (0,0)} \alpha_2(h_1, h_2) = 0.$$

Proof of SCD Theorem continued

A similar expression exists for the function $v(x, y)$ near (x_0, y_0) .

$$\begin{aligned}v(x_0 + h_1, y_0 + h_2) &= v(x_0, y_0) + v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2 \\ &\quad + \alpha_3(h_1, h_2)h_1 + \alpha_4(h_1, h_2)h_2\end{aligned}$$

where α_3 and α_4 satisfy

$$\lim_{(h_1, h_2) \rightarrow (0,0)} \alpha_3(h_1, h_2) = 0 \quad \text{and} \quad \lim_{(h_1, h_2) \rightarrow (0,0)} \alpha_4(h_1, h_2) = 0.$$

Step 1: Let $\epsilon > 0$ be given. Since

$$\lim_{(h_1, h_2) \rightarrow (0,0)} \alpha_j(h_1, h_2) = 0 \quad \forall j \in \{1, 2, 3, 4\},$$

there exists a $\delta > 0$ such that, for each $j \in \{1, 2, 3, 4\}$,

$$|\alpha_j(h_1, h_2)| < \epsilon/4 \text{ whenever } 0 < \sqrt{h_1^2 + h_2^2} < \delta.$$

Proof of SCD Theorem continued

Step 2: Expand the numerator of the difference quotient into real and imaginary parts. We compute that $f(z_0 + h) - f(z_0)$

$$\begin{aligned} &= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) + i[v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)] \\ &= u_x(x_0, y_0)h_1 + u_y(x_0, y_0)h_2 + i[v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2] \\ &\quad + \alpha_1(h_1, h_2)h_1 + \alpha_2(h_1, h_2)h_2 + i[\alpha_3(h_1, h_2)h_1 + \alpha_4(h_1, h_2)h_2] \\ &= u_x(x_0, y_0)[h_1 + i h_2] + i v_x(x_0, y_0)[h_1 + i h_2] \quad (\text{Cauchy-Riemann}) \\ &\quad + [\alpha_1(h_1, h_2) + i \alpha_3(h_1, h_2)]h_1 + [\alpha_2(h_1, h_2) + i \alpha_4(h_1, h_2)]h_2 \end{aligned}$$

Therefore, we have that $\frac{f(z_0+h)-f(z_0)}{h}$ is equivalent to

$$u_x(x_0, y_0) + i v_x(x_0, y_0) + (\alpha_1 + i \alpha_3) \frac{h_1}{h} + (\alpha_2 + i \alpha_4) \frac{h_2}{h}.$$

Proof of SCD Theorem continued

Step 3: Bound the “error term.” Using the fact that $|h_1/h| \leq 1$ and $|h_2/h| \leq 1$, we have that $\left| \frac{f(z_0 + h) - f(z_0)}{h} - [u_x(x_0, y_0) + i v_x(x_0, y_0)] \right|$

$$= \left| (\alpha_1 + i\alpha_3) \frac{h_1}{h} + (\alpha_2 + i\alpha_4) \frac{h_2}{h} \right|$$

$$\leq |\alpha_1 + i\alpha_3| \left| \frac{h_1}{h} \right| + |\alpha_2 + i\alpha_4| \left| \frac{h_2}{h} \right|$$

$$\leq |\alpha_1 + i\alpha_3| + |\alpha_2 + i\alpha_4|$$

$$\leq |\alpha_1| + |\alpha_3| + |\alpha_2| + |\alpha_4|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4}$$

$$= \epsilon \quad \text{whenever} \quad 0 < \sqrt{h_1^2 + h_2^2} = |h| < \delta. \quad \text{QED}$$

An Important Example

Example: (In-class exercise)

Let

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}.$$

Show that $f'(z)$ exists for all $z \in \mathbb{C}$ and find a formula for $f'(z)$.