MATH 305 Complex Analysis, Spring 2016

Important Functions and Their Properties

Worksheet for Chapter 3

The following are some key complex functions along with their significant properties. This material, some of which has already been discussed in class, is covered in Chapter 3 of the course text. You should read the worksheet carefully and complete the exercises.

The Exponential Function

$$e^z \;=\; e^{x+iy} \;=\; e^x \cdot e^{iy} \;=\; e^x(\cos y + i \sin y) \;=\; e^x \cos y + i e^x \sin y$$

1. $|e^z| = e^x$ and $\arg(e^z) = y + 2\pi n, n \in \mathbb{Z}$
2. $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$
3. $\frac{d}{dz}\left(e^z\right) = e^z$
4. e^z is **periodic** with period $2\pi i$, that is $e^{z+2\pi i} \;=\; e^z \quad \forall\, z \in \mathbb{C}$.
5. The range of e^z is $\mathbb{C} - \{0\}$.

The Logarithmic Function

$$\log z \;=\; \ln|z| + i \arg(z) \;=\; \ln r + i(\theta + 2\pi n),\; n \in \mathbb{Z} \quad (\text{where } z = re^{i\theta})$$

$$\operatorname{Log} z \;=\; \ln|z| + i \operatorname{Arg}(z) \;=\; \ln r + i\,\theta,\; (\text{where } z = re^{i\theta} \text{ and } -\pi < \theta \le \pi)$$

1. $\log z$ is a **multiple-valued function** while $\operatorname{Log} z$, the **principal value** of $\log z$, is single valued.
2. $e^{\log z} = z$ for **any** value of $\log z$.
3. The domain of $\log z$ and $\operatorname{Log} z$ is $\mathbb{C} - \{0\}$.
4. $\log(e^z) = z + i\,2\pi n,\; n \in \mathbb{Z}$
5. If $z = x + iy$ is chosen so that $-\pi < y \le \pi$, then $\operatorname{Log}(e^z) = z$ (see Exercise #8 on p. 97).
6. For **some** value of each logarithm,
   - $\log(z_1 z_2) = \log z_1 + \log z_2$
   - $\log\left(\dfrac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$
   - $\log(z^n) = n \log z,\; n \in \mathbb{Z}$,

   but these properties do not hold generally.
7. $z^n = e^{n \log z},\; n \in \mathbb{Z}$ holds for **all** values of $\log z$.
8. $z^{1/n} = e^{\frac{1}{n}\log z},\; n \in \mathbb{Z} - \{0\}$

Exercise 0.1 *Let* $z = 3 - 3\sqrt{3}\, i$.

(a) *Compute* $\log z$ *and* $\text{Log } z$.

(b) *Compute* $e^{\log z}$ *and* $\text{Log}(e^z)$. *Conclude that* $e^{\log z} = z$, *but in this case,* $\text{Log } e^z \neq z$.

Exercise 0.2 *Verify that* $z^n = e^{n \log z}$, $n \in \mathbb{Z}$ *is true for all values of* $\log z$. *Start by writing* $z = re^{i\theta}$.

Branches of the Logarithmic Function

$$\log z = \ln r + i\theta, \quad \text{where } z = re^{i\theta} \text{ and } \alpha < \theta < \alpha + 2\pi,\ \alpha \in \mathbb{R} \text{ a fixed angle}$$

$$\text{Log } z = \ln r + i\theta, \quad \text{where } z = re^{i\theta} \text{ and } -\pi < \theta < \pi$$

1. In general, a **branch** of a multiple-valued function is a choice or restriction that makes the function single-valued and analytic on the given domain. The given definition of $\log z$ above is analytic on the domain $\mathbb{C} - \{re^{i\alpha} : r \geq 0\}$, which is the complex plane minus the ray at angle α. Recall that we must delete this ray in order to make the logarithm continuous. The ray $\theta = \alpha$ is called a **branch cut**.

2. As defined above, $\text{Log } z$ is known as the **principle branch** of $\log z$. Its domain is the complex plane minus the negative real axis.

3. Using the polar form of the Cauchy-Riemann equations, it is straight-forward to show that $\frac{d}{dz}(\log z) = \frac{1}{z}$ and $\frac{d}{dz}(\text{Log } z) = \frac{1}{z}$ on their domains of definition.

Complex Exponents

The fact that $z^n = e^{n \log z}$, $n \in \mathbb{Z}$ motivates the following definition for complex exponents.

Definition 0.3 *For any $z \neq 0$ and for any $c \in \mathbb{C}$, the multiple-valued function z^c (sometimes single-valued) is defined by*

$$z^c \;=\; e^{c \log z},$$

where $\log z = \ln|z| + i \arg(z)$ is the multiple-valued logarithm. The **principal value** *of z^c is*

$$z^c \;=\; e^{c \operatorname{Log} z},$$

where $\operatorname{Log} z$ is the principal value of the logarithm function.

Note: For each $n \in \mathbb{Z}$, $z^n = e^{n \log z}$ outputs one value, the usual value of the power function. In other words, the usual definitions of $z, z^{-1}, z^2, z^{-2}, \ldots$ are unchanged by this new definition. Similarly, $z^{1/n}$ still outputs the n roots of the complex number z. Otherwise, there are an infinite number of values of z^c unless a particular branch or value of the logarithmic function is specified.

Exercise 0.4 *Show that $(-2)^i = e^{-\pi(2n+1)}[\cos(\ln 2) + i \sin(\ln 2)]$, $n \in \mathbb{Z}$. Conclude that the values of $(-2)^i$ all lie on the ray $\theta = \ln 2 \approx 40^\circ$. What is the principal value of $(-2)^i$?*

Branches of z^c

Since the definition of z^c depends on $\log z$, we must choose a branch of $\log z$ in order to make z^c an analytic function. The natural choice is to restrict the logarithm by deleting the ray $\theta = \alpha$ for some fixed $\alpha \in \mathbb{R}$.

Theorem 0.5 *For a fixed $\alpha \in \mathbb{R}$, the branch of the function z^c defined as*

$$z^c \;=\; e^{c \log z}, \quad \text{where } \log z \;=\; \ln r + i\theta, \; \alpha < \theta < \alpha + 2\pi,$$

is analytic on its domain and $\frac{d}{dz}(z^c) = cz^{c-1}$. (The Power Rule holds for any complex exponent!)

Proof: With the given branch and branch cut for $\log z$, we know that $\log z$ is analytic and that $\frac{d}{dz}(\log z) = \frac{1}{z}$. Since e^z is analytic on all of $\mathbb{C}$, z^c is analytic on its domain since it is the composition of two analytic functions (chain rule). We have

$$\begin{aligned}
\frac{d}{dz}(z^c) &= e^{c \log z} \cdot c \cdot \frac{1}{z} && \text{(chain rule)} \\
&= c \cdot \frac{e^{c \log z}}{e^{\log z}} && \text{since } e^{\log z} = z \text{ holds } \forall\, z \in \mathbb{C} \\
&= c \cdot e^{c \log z - \log z} && \text{by property 2 of } e^z \\
&= c \cdot e^{(c-1) \log z} && \text{factor out the } \log z \\
&= cz^{c-1} && \text{using the definition of } z^c. \qquad \textit{QED}
\end{aligned}$$

For example, if $f(z) = z^i$ is defined appropriately, then $f'(z) = iz^{i-1}$.

Trigonometric Functions

$$\cos z \;=\; \frac{e^{iz}+e^{-iz}}{2} \qquad \text{and} \qquad \sin z \;=\; \frac{e^{iz}-e^{-iz}}{2i} \qquad \forall\, z \in \mathbb{C}$$

The remaining four trig functions are defined as usual: $\tan z = \dfrac{\sin z}{\cos z}$, $\sec z = \dfrac{1}{\cos z}$, etc.

1. The definitions for $\cos z$ and $\sin z$ are inspired by Euler's formula. Replacing z by θ in each of the above formulas and using $e^{i\theta} = \cos\theta + i\sin\theta$ gives true statements.

2. Both $\cos z$ and $\sin z$ are entire functions (analytic on all of $\mathbb{C}$) since e^{iz} and e^{-iz} are entire.

3. As expected, $\frac{d}{dz}(\sin z) = \cos z$ and $\frac{d}{dz}(\cos z) = -\sin z$.

4. $\cos(-z) = \cos z$ (even function) and $\sin(-z) = -\sin z$ (odd function).

5. Euler's formula holds for **any** complex number:

$$e^{iz} \;=\; \cos z + i\sin z \quad \forall\, z \in \mathbb{C}.$$

6. Most of the usual identities hold true:

$$\cos^2 z + \sin^2 z \;=\; 1$$

$$\begin{aligned} \sin(z_1+z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\ \cos(z_1+z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \end{aligned}$$

7. Both $\sin z$ and $\cos z$ are periodic of period 2π, that is, $\sin(z+2\pi) = \sin z$ and $\cos(z+2\pi) = \cos z$.

8. If $z = x+iy$, then

$$\begin{aligned} \cos z &= \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \sin x \cosh y + i \cos x \sinh y, \end{aligned}$$

   where $\cosh y = (e^y + e^{-y})/2$ is the hyperbolic cosine function and $\sinh y = (e^y - e^{-y})/2$ is hyperbolic sine.

9. $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$. This implies that, unlike the real case, both functions are unbounded since $\lim\limits_{y\to\infty} \sinh y = \infty$.

Exercise 0.6 *Compute* $\cos(i)$ *and* $\sin(i)$ *and verify that* $\cos^2(i) + \sin^2(i) = 1$.

Exercise 0.7 *Using the definition of each function, verify that* $\frac{d}{dz}(\sin z) = \cos z$ *and* $\frac{d}{dz}(\cos z) = -\sin z$.

Exercise 0.8 *Explain why* $e^{iz} = \cos z + i \sin z$ *holds for any complex number* z.

Exercise 0.9 *If* $z = x + iy$, *show that* $\cos z = \cos x \cosh y - i \sin x \sinh y$ *and that* $|\cos z|^2 = \cos^2 x + \sinh^2 y$.