MATH 305 Complex Analysis, Spring 2016

Using Residues to Evaluate Improper Integrals

Worksheet for Sections 78 and 79

One of the interesting applications of Cauchy's Residue Theorem is to find exact values of *real* improper integrals. The idea is to integrate a complex rational function around a closed contour C that can be arbitrarily large. As the size of the contour becomes infinite, the piece in the complex plane (typically an arc of a circle) contributes 0 to the integral, while the part remaining covers the entire real axis (e.g., an improper integral from $-\infty$ to ∞).

An Example

Let us use residues to derive the formula

$$\int_0^\infty \frac{x^2}{x^4 + 1}\, dx = \frac{\sqrt{2}\,\pi}{4}. \tag{1}$$

Note the somewhat surprising appearance of π for the value of this integral.

First, let $f(z) = \dfrac{z^2}{z^4 + 1}$ and let $C = L_R + C_R$ be the contour that consists of the line segment L_R on the real axis from $-R$ to R, followed by the semi-circle C_R of radius R traversed CCW (see figure below). Note that C is a positively oriented, simple, closed contour. We will assume that $R > 1$.
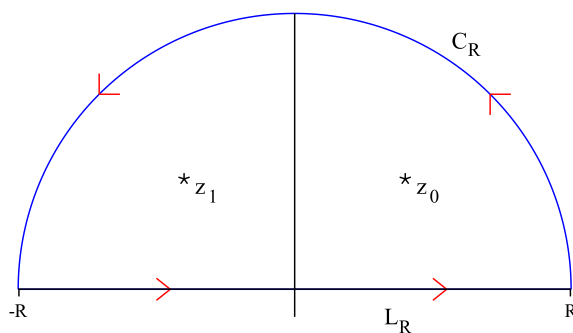
Next, notice that $f(z)$ has two singular points (simple poles) inside C. Call them z_0 and z_1, as shown in the figure. By Cauchy's Residue Theorem. we have

$$\oint_C f(z)\, dz = 2\pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right)$$

On the other hand, we can parametrize the line segment L_R by $z = x, -R \le x \le R$, so that

$$\oint_C f(z)\, dz = \int_{-R}^R \frac{x^2}{x^4 + 1}\, dx + \int_{C_R} \frac{z^2}{z^4 + 1}\, dz,$$

since $C = L_R + C_R$. Moreover, since f is an even function, the middle integral in the above expression can be written as $2 \displaystyle\int_0^R \frac{x^2}{x^4 + 1}\, dx$. Putting this all together, we find that

$$2\pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right) \;=\; 2 \int_0^R \frac{x^2}{x^4+1}\, dx \;+\; \int_{C_R} \frac{z^2}{z^4+1}\, dz. \tag{2}$$

Equation (2) is valid for any $R > 1$. Thus, we can take the limit as $R \to \infty$ of both sides. Using the "ML-Theorem," it is possible to show that the integral over the semi-circle goes to 0 as $R \to \infty$. Consequently,

$$\int_0^\infty \frac{x^2}{x^4+1}\, dx \;=\; \pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right). \tag{3}$$

All that remains is to find the residues of f at z_0 and z_1.

There is a neat trick that can be used to find the residue of a simple pole. Suppose that $f(z) = p(z)/q(z)$ is the ratio of two polynomials p and q, and that α is a simple root of q. This means that $q(\alpha) = 0$, but $q'(\alpha) \neq 0$. Since α is a root of q, we can write $q(z) = (z-\alpha)S(z)$, where S is a remainder polynomial of degree one less than q. Using the product rule, we have

$$q'(z) \;=\; S(z) + (z-\alpha)S'(z) \quad \implies \quad q'(\alpha) \;=\; S(\alpha).$$

To find the residue of f at the simple pole $z = \alpha$, we use the theorem from Section 73. Since

$$f(z) \;=\; \frac{p(z)}{q(z)} \;=\; \frac{p(z)}{(z-\alpha)S(z)},$$

we see that $\phi(z) = p(z)/S(z)$ is the analytic portion of $f(z)$ that does not contain the pole α. Consequently,

$$\operatorname*{Res}_{z=\alpha} f(z) \;=\; \operatorname*{Res}_{z=\alpha} \frac{p(z)}{q(z)} \;=\; \frac{p(\alpha)}{S(\alpha)} \;=\; \frac{p(\alpha)}{q'(\alpha)}. \tag{4}$$

In other words, the residue may be found by evaluating the numerator at the pole divided by the *derivative* of the denominator at the pole. This is typically easier than doing a partial fraction decomposition, but it only works for poles of order 1.

Exercises:

1. **(a)** Find the precise values of the two singular points z_0 and z_1 for $f(z) = \dfrac{z^2}{z^4+1}$.

    (b) Use the "ML-Theorem" and the triangle inequality to show that

$$\left| \int_{C_R} \frac{z^2}{z^4+1}\, dz \right| \;\leq\; \frac{\pi R^3}{R^4-1}.$$

Conclude that the integral of $f(z)$ over the semi-circle C_R goes to 0 as $R \to \infty$.

(c) Use formula (4) to compute the residues of $f(z)$ at z_0 and z_1. Plug these values into equation (3) to finish the derivation of formula (1).

2. Using the same approach as the previous example, find the value (with proof) of

$$\int_0^\infty \frac{1}{(x^2+1)^2}\, dx.$$

3. **Extra Credit:** Show that
$$\int_0^\infty \frac{1}{x^3+1}\, dx = \frac{2\pi}{3\sqrt{3}}.$$

Note: See Figure 95 on p. 268 of the text for the right contour to use.