MATH 305 Complex Analysis, Spring 2016

Basic Algebraic Properties of Complex Numbers

The following are some of the key algebraic properties of complex numbers. They are covered in Sections 2 and 3 of the course text. We will prove a few of these in class while you will prove some others for homework. Many of the proofs follow from the corresponding properties in the real case.

Punchline: Almost all of the standard algebraic rules and axioms for real numbers hold for complex numbers as well.

1. **Commutative:** $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$ holds for any $z_1, z_2 \in \mathbb{C}$.

2. **Associative:** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ holds for any $z_1, z_2, z_3 \in \mathbb{C}$.

3. **Distributive:** $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ holds for any $z_1, z_2, z_3 \in \mathbb{C}$.

4. **Additive Identity:** $z + 0 = z$ for any $z \in \mathbb{C}$. 0 is the only complex number with this property (uniqueness).

5. **Multiplicative Identity:** $z \cdot 1 = z$ for any $z \in \mathbb{C}$. 1 is the only complex number with this property (uniqueness).

6. **Additive Inverse:** For each $z \in \mathbb{C}$, there exists an additive inverse $-z = -x + i(-y) = -x + -yi$ satisfying $z + -z = 0$.

7. **Multiplicative Inverse:** For each nonzero $z \in \mathbb{C}$, there exists a multiplicative inverse z^{-1} satisfying $z \cdot z^{-1} = z^{-1} \cdot z = 1$. If $z = x + iy$, then

   $$z^{-1} \;=\; \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}.$$

   This formula can be derived using the **complex conjugate** $\overline{z} = x - iy$. Note that z^{-1} exists as long as $x^2 + y^2 \neq 0$, which is satisfied as long as both x and y do not vanish. In other words, z^{-1} exists for all $z \in \mathbb{C} - \{0\}$.

8. **Zero Property:** If $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

9. **Subtraction:** $z_1 - z_2 = z_1 + (-z_2)$ (add the additive inverse of z_2 to z_1).

   If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$ (subtract the real and imaginary parts separately).

10. **Division:** $\dfrac{z_1}{z_2} = z_1 \cdot z_2^{-1}$. In other words, dividing by the complex number z_2 (assuming it is nonzero) is equivalent to multiplying by the multiplicative inverse of z_2. In practice, this is computed by multiplying top and bottom by the complex conjugate. For instance,

    $$\frac{3-2i}{4+i} \;=\; \frac{3-2i}{4+i} \cdot \frac{4-i}{4-i} \;=\; \frac{10-11i}{17} \;=\; \frac{10}{17} - \frac{11}{17}i.$$

    Note that if $z_1 = 1$, then we have the expected identity

    $$\frac{1}{z_2} \;=\; z_2^{-1}.$$

11. **Integer Exponents:** For any integers m and n,

   (a) $z^m \cdot z^n = z^{m+n}$

   (b) $z^{-m} = \dfrac{1}{z^m}$

   (c) $(z^m)^n = z^{mn}$

   (d) $(z_1 z_2)^m = z_1^m \cdot z_2^m$

   Note that setting $m = -1$ in the last property gives $(z_1 z_2)^{-1} = z_1^{-1} \cdot z_2^{-1}$. In other words, the inverse of a product is the product of the inverses. This allows us to multiply fractions in the usual way:

$$\frac{z_1}{z_3} \cdot \frac{z_2}{z_4} = \frac{z_1 z_2}{z_3 z_4}.$$