MATH 305, Fall 2011

Computer Lab #1

Mappings of Complex Functions

DUE DATE: Wednesday, September 21, Start of class

The goal for this lab project is to use the computer software package Maple to visualize mappings of complex functions. In addition to using commands in Maple, you will also need to perform some calculations by hand in order to understand the graphs shown on the computer screen.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, website, another student, etc. should all be appropriately referenced. Please turn in **one report per group**, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, writing in complete sentences. Your report should be TYPED and you are encouraged to type all of it in your Maple worksheet. Calculations can be attached in an appendix or written out amidst your report.

Getting Started

We will be using some graphics commands in the `plots` package. Start by typing the command

```
with(plots):
```

Recall that a colon : suppresses the output of a Maple command while a semi-colon `;` will display the output. In the latter case, you will see a list of available commands in the `plots` package. You will need to run this command every time you start a new session of Maple.

Maple includes a feature that allows you to plot the images of a rectangular grid under a complex function $f(z)$. We will use this command to visualize the image of the grid under a particular function $f(z)$. The general form of this command is

```
conformal(f(z), z = a+Ib..c+Id, scaling = constrained, grid = [n,m]);
```

Note that the imaginary number i is written as `I` in Maple. The complex numbers `a+Ib` and `c+Id` form the lower left and upper right corners of the rectangular grid, respectively. The option `grid = [n,m]` allows you to specify the number of gridlines in the vertical and horizontal directions, respectively.

Let's begin with a rectangular grid with corners at $-1-i$ and $1+i$ and gridlines spaced at intervals of 0.25 units. To plot the rectangular grid itself, use the following command:

```
conformal(z, z = -1-I .. 1+I, scaling = constrained, grid = [9, 9])
```

You should see a plot containing 9 horizontal and 9 vertical lines over the range $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Next, plot the image of this grid in the w-plane under the mapping $w = f(z) = z^2$. To do this, use the command:

```
conformal(z^2, z = -1-I .. 1+I, scaling = constrained, grid = [9, 9])
```

Another useful way in which to visualize a complex mapping is to plot the image of a particular curve (e.g., circle) in the w-plane. This can be accomplished in Maple by plotting a **parametric curve**. Suppose we have a curve C in the z-plane described parametrically as $(x(t), y(t))$, where x and y are each functions of the time parameter t, and as t varies over an interval J, the curve C is traced out in the z-plane. Given a complex function $w = f(z) = u(x, y) + iv(x, y)$, we can compute the image of C in the w-plane as

$$(u(x(t), y(t)),\ v(x(t), y(t))).$$

As t varies over the interval J, this parametric expression will trace out the image of C in the w-plane. The general form for plotting a parametric curve in Maple is

```
plot([x(t), y(t), t = a .. b], scaling = constrained);
```

For example, the command

```
plot([cos(t), sin(t), t = -2*Pi .. 2*Pi], scaling = constrained);
```

traces out the unit circle twice in the z-plane.

In addition to plotting a parametric equation, it is possible to animate a parametric curve using the Maple command `animatecurve`. For example, executing

```
animatecurve([cos(t), sin(t), t = -2*Pi .. 2*Pi], frames = 30, numpoints = 75)
```

will provide a set of coordinate axis with no graph. However, clicking on the graph with the mouse will then give a DVD style play/pause control panel at the top of the screen. You can use this to animate the curve. Try pressing play to see the curve animated. The `frames=30` command gives the number of frames viewed throughout the total animation. Increasing this number will give more frames to your "movie" but it will take longer to watch the entire process unfold. The `numpoints=75` command controls the number of points plotted when producing the curve. Higher numbers yield more accurate plots but take longer to produce. Note that the `animatecurve` command needs to be loaded with the package `plots` each time you start a new Maple session.

Exercises:

1. Consider the function $f(z) = z^2$. In class we showed that this can be written as $w = u(x, y) + iv(x, y)$ where $u = x^2 - y^2$ and $v = 2xy$. For this problem, use a rectangular grid with corners at $-1 - i$ and $1 + i$ and `grid=[9,9]`.

   a. Use Maple to plot the image of the rectangular grid under $f(z) = z^2$. What type of curves appear in the w-plane?

   b. Suppose that we set $y = c$, where c is some nonzero real constant, and look for the image under f of this horizontal line. Show that the image satisfies the equation

   $$u = \frac{v^2}{4c^2} - c^2. \tag{1}$$

   This equation describes what curve in the uv-plane? Which way does it open? What is its vertex?

c. Next, suppose that we set $x = d$, where d is some nonzero real constant, and look for the image under f of this vertical line. Find an equation similar to equation (1) that describes the image. What curve in the uv-plane does your equation describe? Which way does it open? What is its vertex?

d. What is the image of the line $y = 0$? What is the image of the line $x = 0$?

e. Pick any point, other than the origin, where the image of two gridlines cross. Find the slopes of the two curves intersecting at that point and show that they cross at right angles. Label this point on your graph (and turn it in with your lab report.)

f. Explain why there are fewer curves in the image of the grid than in the original grid itself. *Hint:* Why are the image curves darker?

2. Plot the image of the same grid used in question 1 under the mapping $w = z^3$. Although the resulting equations for the images of the horizontal and vertical grid lines are more complicated algebraically, we can still understand some of the features without trying to find an equation between u and v. For example, notice that the image of one of the vertical lines has a double point (crosses itself) at the point $w = -1$. Why does this double point occur? Which vertical line is mapped to this curve?

3. Use the `plot` command to draw a parametric plot of the image of the unit circle under the mapping $w = f(z) = z^2 - z$. You should see another double point at $w = -1$. Find the two points on the unit circle that are mapped to this double point. The `animate` command may be useful here.

4. Consider the function $f(z) = e^z$. In class we showed that this can be written in polar form as $w = \rho e^{i\phi}$ where $\rho = e^x$ and $\phi = y$. For this problem, use a rectangular grid with corners at $-1 - \frac{\pi}{2}i$ and $1 + \frac{\pi}{2}i$ and `grid=[11,11]`.

a. Use Maple to plot the image of the rectangular grid under $f(z) = e^z$. You will need to type `exp(z)` for e^z. What type of curves appear in the w-plane?

b. Suppose that we set $y = c$, where c is some real constant, and look for the image under f of this horizontal line. What type of curve do you obtain? Explain. Be as precise as possible. For your plot, what is the angle between each successive image of a line of the form $y = c$?

c. Next, suppose that we set $x = d$, where d is some real constant, and look for the image under f of this vertical line. What type of curve do you obtain? Explain. Be as precise as possible. What happens when you expand the grid to have corners $-1 - \pi i$ and $1 + \pi i$? Why?

d. At what angle does the image of a horizontal gridline meet the image of a vertical gridline?

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