Section 22: Sufficient Conditions for Differentiability

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Class Notes for MATH 305, Complex Analysis
September 26, 2011

Cauchy-Riemann Equations

Recall: The Cauchy-Riemann Equations must be satisfied in order for a complex function $f(z) = u(x, y) + i\,v(x, y)$ to be differentiable at a point.

$$u_x = v_y, \quad u_y = -v_x$$

They are derived by evaluating the limit definition of the derivative approaching in the real direction and the pure imaginary direction, and then equating the two results.

Let $z_0 = x_0 + i\,y_0$. If $f'(z_0)$ exists, then it's value is $u_x + i\,v_x$, where each partial is evaluated at (x_0, y_0).

Important: The Cauchy-Riemann equations are **necessary** conditions for $f'(z_0)$ to exist, but they are not sufficient.

Main Theorem in the Section

Theorem (SCD – Sufficient Conditions for Differentiability)

Suppose that $f(z) = u(x, y) + i\, v(x, y)$ *is defined in a neighborhood of* $z_0 = x_0 + i\, y_0$ *and that*

1. u_x, u_y, v_x, v_y *exist everywhere in the neighborhood,*
2. $u_x = v_y, u_y = -v_x$ *at* (x_0, y_0)*, and*
3. u_x, u_y, v_x, v_y *are* **continuous** *at* (x_0, y_0).

Then $f'(z_0)$ *exists and* $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$.

Note: It is possible to satisfy the Cauchy-Riemann equations at a point, yet **not** be differentiable there (see Exercise #6 in Section 23 for such an example – HW). The point of the theorem is that continuity of the partial derivatives, not just satisfying the Cauchy-Riemann equations, is also required to insure that the derivative $f'(z_0)$ exists.

Proof of SCD Theorem

Proof: Assume that the hypotheses of the theorem are true. We will show that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \;=\; u_x(x_0, y_0) + i\, v_x(x_0, y_0),$$

thereby proving that the derivative exists and the value is the one given in the theorem.

The proof relies on the multivariable version of Taylor's Theorem. If both partial derivatives of a function $u(x, y)$ exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then we can evaluate u nearby using the expression

$$\begin{aligned} u(x_0 + h_1, y_0 + h_2) \;&=\; u(x_0, y_0) \;+\; u_x(x_0, y_0)h_1 \;+\; u_y(x_0, y_0)h_2 \\ &+\; \alpha_1(h_1, h_2)h_1 + \alpha_2(h_1, h_2)h_2 \end{aligned}$$

where α_1 and α_2 satisfy

$$\lim_{(h_1,h_2)\to(0,0)} \alpha_1(h_1, h_2) \;=\; 0 \quad \text{and} \quad \lim_{(h_1,h_2)\to(0,0)} \alpha_2(h_1, h_2) \;=\; 0.$$

Proof of SCD Theorem continued

A similar expression exists for the function $v(x, y)$ near (x_0, y_0).

$$\begin{aligned} v(x_0 + h_1, y_0 + h_2) &= v(x_0, y_0) + v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2 \\ &+ \alpha_3(h_1, h_2)h_1 + \alpha_4(h_1, h_2)h_2 \end{aligned}$$

where α_3 and α_4 satisfy

$$\lim_{(h_1,h_2)\to(0,0)} \alpha_3(h_1, h_2) = 0 \quad \text{and} \quad \lim_{(h_1,h_2)\to(0,0)} \alpha_4(h_1, h_2) = 0.$$

Step 1: Let $\epsilon > 0$ be given. Since

$$\lim_{(h_1,h_2)\to(0,0)} \alpha_j(h_1, h_2) = 0 \quad \forall j \in \{1, 2, 3, 4\},$$

there exists a $\delta > 0$ such that, for each $j \in \{1, 2, 3, 4\}$,

$$|\alpha_j(h_1, h_2)| < \epsilon/4 \text{ whenever } 0 < \sqrt{h_1^2 + h_2^2} < \delta.$$

Proof of SCD Theorem continued

Step 2: Expand the numerator of the difference quotient into real and imaginary parts. We compute that $f(z_0 + h) - f(z_0)$

$$\begin{aligned}
&= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) + i[v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)] \\
&= u_x(x_0, y_0)h_1 + u_y(x_0, y_0)h_2 + i[v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2] \\
&\quad + \alpha_1(h_1, h_2)h_1 + \alpha_2(h_1, h_2)h_2 + i[\alpha_3(h_1, h_2)h_1 + \alpha_4(h_1, h_2)h_2] \\
&= u_x(x_0, y_0)[h_1 + i\, h_2] + i\, v_x(x_0, y_0)[h_1 + i\, h_2] \ \text{(Cauchy-Riemann)} \\
&\quad + [\alpha_1(h_1, h_2) + i\, \alpha_3(h_1, h_2)]h_1 + [\alpha_2(h_1, h_2) + i\, \alpha_4(h_1, h_2)]h_2
\end{aligned}$$

Therefore, we have that $\frac{f(z_0+h)-f(z_0)}{h}$ is equivalent to

$$u_x(x_0, y_0) + i\, v_x(x_0, y_0) + (\alpha_1 + i\, \alpha_3)\frac{h_1}{h} + (\alpha_2 + i\, \alpha_4)\frac{h_2}{h}.$$

Step 3: Bound the "error term." Using the fact that $|h_1/h| \leq 1$ and $|h_2/h| \leq 1$, we have that $\left| \frac{f(z_0+h) - f(z_0)}{h} - [u_x(x_0,y_0) + i\, v_x(x_0,y_0)] \right|$

$$
\begin{aligned}
&= \left| (\alpha_1 + i\,\alpha_3)\frac{h_1}{h} + (\alpha_2 + i\,\alpha_4)\frac{h_2}{h} \right| \\
&\leq |\alpha_1 + i\,\alpha_3| \left|\frac{h_1}{h}\right| + |\alpha_2 + i\,\alpha_4| \left|\frac{h_2}{h}\right| \\
&\leq |\alpha_1 + i\,\alpha_3| \; + \; |\alpha_2 + i\,\alpha_4| \\
&\leq |\alpha_1| + |\alpha_3| + |\alpha_2| + |\alpha_4| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= \epsilon \quad \text{whenever} \quad 0 < \sqrt{h_1^2 + h_2^2} = |h| < \delta. \qquad QED
\end{aligned}
$$

An Important Example

Example: (In-class exercise)
Let

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}.$$

Show that $f'(z)$ exists for all $z \in \mathbb{C}$ and find a formula for $f'(z)$.