MATH 305 Complex Analysis

Sample Final Exam Questions

1. Express each quantity in rectangular form $a + bi$ and polar form $re^{i\theta}$.

   (a) $(1 + 7i)(3 + 4i)$

   (b) $(1 + \sqrt{3}i)^8$

2. Write the cube roots of $-8i$ in polar and rectangular form, and sketch the triangle in the complex plane that has these roots as its vertices.

3. Let R be the region in the complex plane defined by

$$R = \{z \mid 0 \leq \text{Re}(z) \leq 2\}$$

   (a) Sketch the region R in the complex plane.

   (b) Sketch the image of R under the map $g(z) = e^z$.

4. Use the SDC theorem to determine where each of the following functions is differentiable. (You don't need to compute their derivatives.)

   (a) $f(z) = y^2 + ix^2$

   (b) $f(z) = x - 2xy + i(x^2 + y - y^2)$

5. Find a function $v(x, y)$ such that the function $f(z) = (y + e^x \sin y) + i\,v(x, y)$ is entire.

6. **Quickies:**

   (a) If $f(z) = \dfrac{1}{e^z + 1}$, what is the radius of the largest disk about $z_0 = 0$ for which the Taylor series of $f(z)$ is guaranteed to converge?

   (b) State the principal branch of the function $\text{Log}(z)$ and sketch the domain of the function.

   (c) Name the mathematician with more theorems bearing his name than any one else.

   (d) If the Cauchy-Riemann equations are satisfied for the function $f(z)$ at the point z_0, then $f'(z_0)$ exists. This statement is (choose one): TRUE FALSE

   (e) Find and simplify the principal value of $(-i)^{1+2i}$.

7. Evaluate the following limits, if they exist. Show your work, making sure to justify your answers thoroughly.

   (a) $\displaystyle\lim_{z\to\infty} \frac{z^2}{(iz + 1)^2}$

   (b) $\displaystyle\lim_{z\to 0} \left(\frac{\bar{z}}{z}\right)^2$

8. Compute the following contour integrals – use parametrizations for the first three. Simplify your answers.

   (a) $\displaystyle\int_C \bar{z}\,dz$ where C is the line segment from 1 to i.

   (b) $\displaystyle\oint_C \frac{1}{z}\,dz$, where C is the unit circle, oriented clockwise.

   (c) $\displaystyle\oint_C \frac{1}{z^2}\,dz$, where C is the unit circle, oriented counterclockwise.

   (d) $\displaystyle\int_i^{i+2} ze^{z^2}\,dz$

9. Let C be the square with vertices $2+2i$, $-2+2i$, $-2-2i$ and $2-2i$, traversed in the clockwise direction. For each function $f(z)$ below, compute $\displaystyle\oint_C f(z)\,dz$. Be sure to specify what theorem or formula you are using.

   (a) $f(z) = \dfrac{e^z}{z - (1 + \frac{1}{2}\pi i)}$

   (b) $f(z) = \dfrac{e^z}{z - (2 + 3i)}$

   (c) $f(z) = \dfrac{z^3}{[z - (1 + i)]^3}$

   (d) $f(z) = \dfrac{z^3}{(z - 1)(z + i)}$

10. Suppose that f and g are entire functions and that $f(z) = g(1/z)$ for all $z \in \mathbb{C} - \{0\}$. Use Liouville's theorem to prove that f must be constant.

11. Let

$$f(z) = \frac{1}{z(z-3)}.$$

   (a) Find the Laurent series expansion of $f(z)$ in the punctured disk $0 < |z| < 3$.

   (b) Find $\operatorname*{Res}_{z=0} f(z)$.

12. Use residues to compute the real improper integral

$$\int_0^\infty \frac{1}{(x^2+9)^2}\,dx\,.$$

Be sure to justify all the steps taken in your calculation.

13. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, explain why or provide a **counterexample**.

(a) $\text{Log}(\frac{z_1}{z_2}) = \text{Log}(z_1) - \text{Log}(z_2)$ for any $z_1, z_2 \in \mathbb{C}$.

(b) If $f(z)$ is continuous on a domain D and $\oint_C f(z)\, dz = 0$ for every closed contour C in D, then $f(z)$ is analytic in D.

(c) There exists a function $g(z)$ that is analytic on the disk $|z| \leq 2$, satisfies $|g(z)| = 1$ on the circle $|z| = 2$, and has $g(0) = 1 + i$.

(d) The function $g(z) = \dfrac{e^z - 1 - z}{z^2}$ has a removable singularity at $z = 0$.