MATH 305 Complex Analysis

Exam #1 SOLUTIONS **Prof. G. Roberts**

1. [14 pts.] **Quickies:** No explanation required.

(a) A function that is analytic for all $z \in \mathbb{C}$ is called an <u>entire</u> function.

(b) The geometric object that helps complex analysts visualize limits involving infinity is called the <u>Riemann sphere</u>.

(c) Suppose that $f(z) = u(x,y) + i\,v(x,y)$ and that

$$\lim_{z \to 3i-2} f(z) = 4 - i\,.$$

Then,

$$\lim_{(x,y)\to(-2,3)} v(x,y) = \underline{-1}.$$

Answer: This is the CR limit theorem.

(d) If the Cauchy-Riemann equations are satisfied for the function $f(z)$ at the point z_0, then $f'(z_0)$ exists. This statement is FALSE.

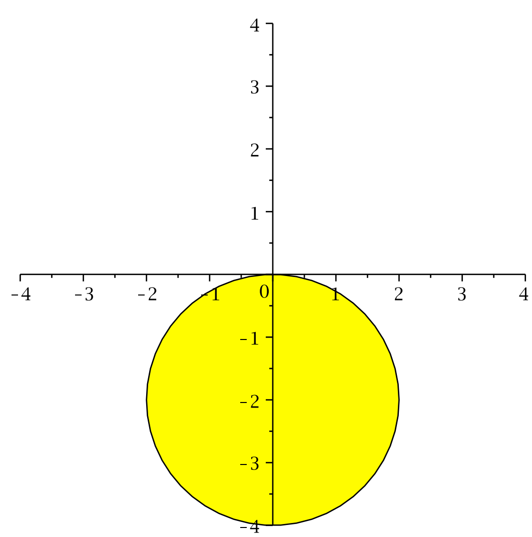
Answer: Just satisfying the Cauchy-Riemann equations alone is not enough to guarantee differentiability. The partial derivatives must also be continuous at z_0.

(e) Sketch the set of points $\{z \in \mathbb{C} : |2\bar{z} - 4i\,| < 4\}$ in the complex plane.

Answer: We compute

$$|2\bar{z} - 4i| = |\overline{2\bar{z} - 4i}| = |2z + 4i| = 2|z + 2i|,$$

so that the inequality simplifies to $|z + 2i| < 2$. This is all points inside the circle of radius 2 centered at $z_0 = -2i$. See figure below – the boundary disk should be dotted since it is not included in the set.



2. [14 pts.] Circle **all** of the following complex numbers that lie on the unit circle. There may be anywhere from zero to six correct choices.

   (a) $-i$

   (b) $\dfrac{2}{3-i}$

   (c) $\sqrt{2}\, e^{i\,\theta}$ for any $\theta \in \mathbb{R}$

   (d) $e^{i\,\sqrt{2}\,\theta}$ for any $\theta \in \mathbb{R}$

   (e) $\dfrac{\overline{z}}{z}$ for any $z \in \mathbb{C} - \{0\}$

   (f) $\cos z + i \sin z$ for any $z \in \mathbb{C}$

   Answer: Choices **(a)**, **(d)**, and **(e)** all lie on the unit circle since they have a modulus of one. For choice **(e)**, note that

$$\left|\frac{\overline{z}}{z}\right| \;=\; \frac{|\,\overline{z}\,|}{|z|} \;=\; \frac{|z|}{|z|} \;=\; 1.$$

   Choice **(b)** has a modulus of $2/\sqrt{10}$, choice **(c)** has a modulus of $\sqrt{2}$ and choice **(f)** has a modulus of e^{-y} where $y = \mathrm{Im}(z)$. This last fact follows from

$$\cos z + i \sin z \;=\; e^{iz} \;=\; e^{i(x+iy)} \;=\; e^{-y} \cdot e^{ix}.$$

3. [10 pts.] Find **all** the roots of $(-2 + 2\sqrt{3}\, i)^{1/4}$ in **rectangular coordinates**. Draw a sketch of the roots in the complex plane.

   Answer: The roots are

$$\pm\sqrt{2}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) \quad \text{and} \quad \pm\,\sqrt{2}\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right).$$

   First convert $-2 + 2\sqrt{3}$ into polar coordinates, obtaining $4e^{i\,\frac{2\pi}{3}}$. Then we have

$$\begin{aligned}
\left(4e^{i\,\frac{2\pi}{3}}\right)^{1/4} &= 4^{1/4} \cdot e^{i\left(\frac{2\pi}{12} + \frac{2\pi k}{4}\right)}, \quad k = 0, 1, 2, 3 \\
&= \sqrt{2}\, e^{i\left(\frac{\pi}{6} + \frac{\pi k}{2}\right)}, \quad k = 0, 1, 2, 3 \\
&= \sqrt{2}\, e^{i\frac{\pi}{6}}, \sqrt{2}\, e^{i\frac{2\pi}{3}}, \sqrt{2}\, e^{i\frac{7\pi}{6}}, \sqrt{2}\, e^{i\frac{5\pi}{3}} \\
&= \sqrt{2}\, e^{i\frac{\pi}{6}}, \sqrt{2}\, e^{i\frac{2\pi}{3}}, -\sqrt{2}\, e^{i\frac{\pi}{6}}, -\sqrt{2}\, e^{i\frac{2\pi}{3}}
\end{aligned}$$

   using the fact that $e^{i\,\pi} = -1$. Converting these polar expressions into rectangular coordinates using Euler's formula gives the answer above. The four roots form a square in the complex plane that is rotated 30° ccw from the positive x-axis and lies on the circle of radius $\sqrt{2}$ centered at the origin.

4. [10 pts.] Compute the following quantities. Be sure to show your work.

(a) Compute the principal value of i^{2+i}.

Answer: We compute

$$i^{2+i} \;=\; e^{(2+i)\cdot \mathrm{Log}(i)} \;=\; e^{(2+i)(\ln|i|+i\,\frac{\pi}{2})} \;=\; e^{(2+i)(i\,\frac{\pi}{2})} \;=\; e^{i\,\pi-\frac{\pi}{2}} \;=\; e^{i\,\pi}\cdot e^{-\frac{\pi}{2}} \;=\; -e^{-\frac{\pi}{2}}.$$

(b) Compute $\log(1-i)$ when the branch $\log z = \ln r + i\,\theta$ ($r>0$, $\frac{3\pi}{2} < \theta < \frac{7\pi}{2}$) is used.

Answer: First note that $\arg(1-i) = -\frac{\pi}{4} + 2\pi n$ with $n \in \mathbb{Z}$. Using the definition of the branch given, we choose $n=1$ and find that $\theta = \frac{7\pi}{4}$ which is the only choice of the argument that lies in the stated interval. Then, we have

$$\log(1-i) \;=\; \ln|1-i| + i\,\frac{7\pi}{4} \;=\; \ln\sqrt{2} + i\,\frac{7\pi}{4} \;=\; \frac{1}{2}\ln 2 + i\,\frac{7\pi}{4}.$$

5. [12 pts.] Evaluate the following limits, if they exist. Show your work, making sure to justify your answers thoroughly.

(a) $\displaystyle\lim_{z\to\infty} \frac{3i(z^2+1)}{(3z-i)^2}$

Answer: Using the LIPI theorem, and later the BLT, we have

$$\begin{aligned}
\lim_{z\to\infty} \frac{3i(z^2+1)}{(3z-i)^2} \;&=\; \lim_{z\to 0} \frac{3i\left(\frac{1}{z^2}+1\right)}{\left(\frac{3}{z}-i\right)^2} \\
&=\; \lim_{z\to 0} \frac{3i(1+z^2)}{(3-i\,z)^2} \\
&=\; \frac{3i\cdot(1+0)}{(3-0)^2} \;=\; \frac{3i}{9} \;=\; \frac{i}{3}.
\end{aligned}$$

(b) $\displaystyle\lim_{z\to 0} \frac{\mathrm{Re}(z)}{z}$

Answer: This limit does not exist, which can be seen by taking the limit in the real and imaginary directions. Suppose $z=x$ and we take the limit along the real axis. Then,

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)}{z} \;=\; \lim_{x\to 0} \frac{x}{x} \;=\; \lim_{x\to 0} 1 \;=\; 1.$$

On the other hand, if z is pure imaginary, $z = i\,y$, we obtain

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)}{z} \;=\; \lim_{y\to 0} \frac{0}{i\,y} \;=\; \lim_{y\to 0} 0 \;=\; 0.$$

Since the two limits do not agree, the limit does not exist.

6. [12 pts.] Consider the function $f(z) = x^2 + 3y^2 + i(2xy + \cos(4x))$.

(a) Find the points in the complex plane (if any) where $f'(z)$ exists and give a formula for the derivative at those points.

Answer: We have $u(x, y) = x^2 + 3y^2$ and $v(x, y) = 2xy + \cos(4x)$. Checking the Cauchy-Riemann equations, we have

$$u_x = 2x = v_y,$$

but $u_y = 6y$ and $v_x = 2y - 4\sin(4x)$. Solving the equation $u_y = -v_x$ yields

$$6y = -2y + 4\sin(4x) \quad \text{or} \quad y = \frac{1}{2}\sin(4x).$$

Since the partial derivatives exist and are continuous on the whole plane, the SCD theorem guarantees that $f'(z)$ exists for all complex numbers $z = x + i\,y$ satisfying $y = \frac{1}{2}\sin(4x)$.

To compute the value of the derivative at these points, we find that

$$f'(z) = u_x + i\,v_x = 2x + i(2y - 4\sin(4x)) = 2x - i\,3\sin(4x) = 2x - i\,6y.$$

(b) Find the points in the complex plane (if any) where $f(z)$ is analytic. Explain.

Answer: The function $f(z)$ is nowhere analytic. Any neighborhood of any point on the curve $y = \frac{1}{2}\sin(4x)$ will contain points not on the curve. Consequently, for any point z_0 on the curve where $f'(z)$ exists, and for any neighborhood of z_0, $f'(z)$ does not exist at **all** points in the neighborhood. By definition, $f(z)$ is nowhere analytic.

7. [12 pts.] Consider the function $g(z) = e^{4+i\,z}$. Show that g is analytic on the entire complex plane in **two** different ways. First, use a well-known theorem about analytic functions. Second, use the SCD (Sufficient Conditions for Differentiability) Theorem.

Answer: First, since both e^z and $4 + i\,z$ (linear function) are analytic on the entire complex plane, so is $g(z)$, as the composition of analytic functions is analytic.

Next, we expand g into real and imaginary parts. We have

$$g(z) = e^{4+i\,z} = e^4 \cdot e^{i\,(x+i\,y)} = e^4 \cdot e^{ix} \cdot e^{-y} = e^{4-y}(\cos x + i\,\sin x),$$

so that $u(x, y) = e^{4-y}\cos x$ and $v(x, y) = e^{4-y}\sin x$. We compute that

$$u_x = -e^{4-y}\sin x = v_y \quad \text{and} \quad u_y = -e^{4-y}\cos x = -v_x$$

so that the Cauchy-Riemann equations are satisfied on all of $\mathbb{C}$. Since the partial derivatives are continuous on the entire plane, the SCD theorem then implies that $f'(z)$ exists for all $z \in \mathbb{C}$.

8. [16 pts.] TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample**.

(a) $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ for any $z_1, z_2 \in \mathbb{C}$.

Answer: FALSE. Take $z_1 = -1 + i$ and $z_2 = -1$. Then

$$\mathrm{Arg}(z_1 z_2) \;=\; \mathrm{Arg}(1 - i) \;=\; -\frac{\pi}{4}$$

but

$$\mathrm{Arg}(z_1) \;=\; \frac{3\pi}{4},\; \mathrm{Arg}(z_2) \;=\; \pi \quad \text{implies} \quad \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \;=\; \frac{7\pi}{4} \neq -\frac{\pi}{4}.$$

(b) If $z \in \mathbb{C} - \{0\}$, then $e^{\log z} = z$ for **any** value of the multiple-valued function $\log z$.

Answer: TRUE. Suppose that θ is a representative for the argument of z. We have

$$\begin{aligned} e^{\log z} &= e^{\ln|z| + i(\theta + 2\pi n)}, \text{ where } n \in \mathbb{Z} \\ &= e^{\ln|z|} \cdot e^{i\,\theta} \cdot e^{i\,2\pi n} \\ &= |z| e^{i\,\theta} \quad \text{since } e^{i\,2\pi n} = 1 \text{ for } n \in \mathbb{Z} \\ &= z. \quad \text{QED} \end{aligned}$$

Note that $z \neq 0$ is important to assume since $\log 0$ is undefined.