MATH 304, Ordinary Differential Equations, Fall 2014

Lab Project #1

Logistic Population Models with Harvesting

DUE DATE: Wednesday, Oct. 1, in class.

The goal of this project is to investigate the effect of harvesting on a logistic population model. You will explore two types of harvesting, constant and periodic. For this lab, p will represent the population of a certain fish species in a particular environment. For example, p might represent the population of Orange Roughy off the coast of Australia, which has been in steady decline due to overfishing and slow growth rates (see p. 406 in the chapter on Australia from *Collapse* by Jared Diamond.) You will make use of certain techniques discussed in class (phase lines, equilibrium points, slope fields and bifurcation diagrams) as well as the software DETools to plot slope fields and approximate solutions to a given ODE. Some questions should be answered by hand, while others require the use of computer software. The ideas for this lab are based on Lab 1.3 in the course text.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Only **one project per group** need be submitted.

Your report should provide coherent answers to the questions below. Be sure to read carefully and answer all of the questions asked. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

1 Constant Harvesting

We have seen that the logistic population model offers a reasonable characterization of a population. For small values of the population, growth is proportional to the size of the population. But as the population gets larger, resource constraints and overcrowding force growth to slow down, even become negative, leading to the idea of a *carrying capacity.* Now suppose we introduce a constant harvesting of the population. For example, we could issue a certain number of fishing licenses that specify how many fish can be caught per year. This would be a direct decrease in the population, modeled by

$$\frac{dp}{dt} \;=\; kp\left(1 - \frac{p}{N}\right) - h$$

where h is the number of fish harvested each year due to fishing and $p(t)$ is the number of fish at time t in years. The positive parameters k and N represent the growth rate and the carrying capacity, respectively. If no fish are caught ($h = 0$), then the model is the standard logistic population model. Notice that our model is an autonomous ODE. The goal is to determine how the fish population is effected as the value of h increases away from 0. The questions below should be answered without using a computer although you may use `HPGSolver` in DETools to help visualize solutions.

1. Find expressions for the equilibrium points of the model in terms of the parameters k, N and h.

2. Thinking of k and N as fixed for the moment, how do the number of equilibria change as h is increased away from $h = 0$? Classify the type of equilibrium points you found in the previous question (source, sink or node). Sketch a graph of the right-hand side of the ODE for different h-values.

3. What is the bifurcation value for this system? Your answer will be in terms of k and N. What type of bifurcation occurs?

4. Sketch the bifurcation diagram (keeping k and N fixed, but treating h as varying) with p on the vertical axis and h on the horizontal axis. Describe the behavior of the fish population for different initial conditions, before, at and after the bifurcation. Be as specific as possible, making sure to include a discussion of the long-term behavior of the fish population.

5. Suppose that fishing has been allowed so that h is slightly larger than the bifurcation value you found in Question 3. A year later, you are in charge of fish management. What recommendations would you give to insure that the fish population survived? Would you continue to allow lots of fishing, just a little fishing or ban it outright? Explain your answer providing evidence from the previous questions.

2 Periodic Harvesting

Next, we amend our model to reflect the fact that harvesting is often periodic. In other words, fishing is usually more productive in one season and less productive in another. We can model this cyclic phenomenon with a standard periodic function like sine or cosine. Consider the model

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{N}\right) - h(1 + \sin(2\pi t))$$

where the parameters have the same meaning as in the previous model except that h is now multiplied by a term varying with time. Since the period of the sine function is one, you can think of the harvesting term as varying over the course of one year. This ODE is no longer autonomous, but it is an example of a *periodic differential equation* because the time-dependent term is a periodic function. Although our usual methods of phase lines and bifurcation diagrams do not apply, keep in mind the results you discovered for the previous model as you answer the questions below.

For the following questions, you should use technology to draw the slope field and plot solutions. You may either use `HPGSolver` in DETools or John Polking's free dfield software (linked from the course homepage). Each program use Java, so be sure to update your computer to the current version. You may need to adjust the window size in your program. In particular, be sure to plot solutions over a **long enough time interval!** To type in the above ODE, use `*` for multiplication, (including before a parentheses), and use `pi` for π.

1. According to the model, what are the maximum and minimum harvesting amounts over the course of the year and when do they occur? You may assume that $t = 0$ corresponds to Jan. 1st. Why is there a $1 + \sin(2\pi t)$ term, rather than just a plain $\sin(2\pi t)$ term?

2. In the model above, set $k = 0.25, N = 4$ and $h = 0.15$. (The values are adjusted so that $N = 4$ might mean 400,000 fish.) What happens to the fish population in this case (long-term behavior) for different initial conditions? Describe the type of solutions obtained. Why does this make sense physically? Print out and turn in one plot showing a few different solutions on the same set of axes.

3. Set $k = 0.25, N = 4$ and $h = 0.25$. What happens to the fish population in this case (long-term behavior) for different initial conditions? Describe the type of solutions obtained. Be sure to plot solutions over a long enough time interval. Print out and turn in one plot showing a few different solutions on the same set of axes.

4. Set $k = 0.25, N = 4$ and $h = 0.26$. What happens to the fish population in this case (long-term behavior) for different initial conditions? Describe the type of solutions obtained. Print out and turn in one plot showing a few different solutions on the same set of axes.

5. Recall that the Poincaré map $P(y_0)$ is defined for a periodic differential equation as the function mapping an initial condition to the point on the solution one period later. In our case, $P(y_0) = y(1)$ where $y(t)$ is the solution to the ODE satisfying the initial condition $y(0) = y_0$. Based on your responses above, what can you say about the number of fixed points for the Poincaré map in each of the three previous questions?

6. **Putting it all together:** What similarities between the two models (constant versus periodic harvesting) do you notice? What is special about the choice of parameter values $k = 0.25$, $N = 4$ and $h = 0.25$ in Question 3? Which model do you think is the more reasonable one for modeling a fish population? Explain.