MATH 304, Ordinary Differential Equations, Fall 2014

Lab Project #3

Investigating a 3D Economic Model

DUE DATE: Friday, Dec. 5, 4:30 pm

The goal of this project is to apply techniques for investigating nonlinear systems (linearization about equilibrium points, bifurcation theory, solution curves, etc.) to an idealized macroeconomic model with foreign capital investment. For this project you will need to do several computations by hand and use MAPLE to visualize solution curves in three dimensions. You will discover some interesting phenomena that occur in the given model.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Only **one project per group** need be submitted.

A Three-dimensional Economic Model

Consider the following autonomous, nonlinear, three-dimensional system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= ay + px(k - y^2) \\ \frac{dy}{dt} &= v(x + z) \\ \frac{dz}{dt} &= mx - ry, \end{aligned}$$

where x, y, z are the dependent variables, and a, p, k, v, m, r are parameters. This system can be viewed as an extension of a famous ODE known as van der Pol's equation. It can be used as an idealized macroeconomic model with foreign capital investment, where x represents the savings of households in a given country, y is the Gross Domestic Product (GDP), and z measures the foreign capital inflow (investment into the country). The six parameters are always considered to be positive. They represent:

$$\begin{aligned} a &= \text{variation of the marginal propensity to savings,} \\ p &= \text{ratio of the capitalized profit,} \\ k &= \text{potential GDP,} \\ v &= \text{output/capital ratio,} \\ m &= \text{capital inflow/savings ratio,} \\ r &= \text{ratio of debt refund to output.} \end{aligned}$$

From an economic perspective, the condition $a > vr$ means that the country's economy is strong enough to refund its debt. In practice, v is usually much smaller than one (it takes large amounts of capital to produce any output), so a stronger, more imposing condition is $a > r$. We will see that the condition $a = r$ is a key bifurcation value, with strong implications for the types of solutions to the model. This system can also be used to model a firm's profits, where x represents reinvestments, y is profit, and z is debt.

Lab Questions

1. Show that the system of differential equations has precisely three equilibrium points:

$$E_0 = (0,0,0), \quad E_1 = (\sqrt{c}, \sqrt{d}, -\sqrt{c}), \quad E_2 = (-\sqrt{c}, -\sqrt{d}, \sqrt{c}),$$

   where

$$c = \frac{amr + kpr^2}{pm^2} \quad \text{and} \quad d = \frac{am + kpr}{pr}.$$

   You should do this calculation by hand. Do any of the equilibrium points make sense in terms of the model? Explain.

2. We would like to linearize the system about each equilibrium point. This will yield a 3×3 matrix in each case. Before we do this, let's explore what happens when we consider linear systems in dimensions greater than two.

   As before, the key qualitative features all depend on the eigenvalues of the matrix, except there are now 3 eigenvalues, as opposed to just 2. Thus, we could have a pair of complex eigenvalues with a positive real part (spiral source) and a negative real eigenvalue (sink) together in the same system (a spiral source/sink?). As before, a real eigenvalue λ and corresponding eigenvector v generate a straight-line solution in the phase space (three-dimensional) given by $\mathbf{Y}(t) = e^{\lambda t} v$. Many of the same concepts from 2×2 systems generalize to higher dimensions. (See Section 3.8 of the course text for more details.)

   Consider the following three matrices and their associated linear systems (b is a parameter):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad C = \begin{bmatrix} -0.1 & 0 & 0 \\ 1 & -1.1 & 0 \\ 2 & 0 & -2.1 \end{bmatrix}.$$

   a. Find the eigenvalues of each matrix.

   b. Use MAPLE to explore the solutions to each system in xyz-space. How do the solutions match up with the eigenvalues? Explain the types of solution curves obtained and why they occur. Turn in a plot for each example showing some sample solution curves. You should try varying the parameter b for small values near 0 (e.g., $b = -0.1$ versus $b = 0.1$). What effect does the sign of b have on the types of solutions in examples A and B?

   c. Find the eigenvectors for matrix C. How do the eigenvectors help explain the graphs of solution curves?

3. Linearize the system about the equilibrium point $E_0 = (0, 0, 0)$.

a. Give the Jacobian matrix and show that the characteristic polynomial is $p(\lambda) = \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$, where $\alpha = -kp, \beta = v(r - a)$, and $\gamma = -v(rpk + am)$.

b. Notice that if $a > r$ is assumed, then all three coefficients α, β, and γ are negative. (Recall that all our parameters are considered to be positive.) Explain why this implies that there is at least one positive eigenvalue for the associated linear system. What does this mean for a typical solution curve of the full system if it begins near the origin?

4. Linearize the system about the equilibrium point $E_1 = (\sqrt{c}, \sqrt{d}, -\sqrt{c})$. (The equilibrium point E_2 gives the same linearization.)

a. Give the Jacobian matrix and show that the characteristic polynomial is $p(\lambda) = \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$, where

$$\begin{aligned} \alpha &= \frac{am}{r}, \\ \beta &= v\left(r + a + \frac{2rpk}{m}\right), \\ \gamma &= 2v(am + rpk). \end{aligned}$$

b. Notice that in this case, all three coefficients α, β, and γ are positive. While it is difficult to apply the formula for the roots of a cubic, it is possible to show that in this case, all of the roots of $p(\lambda)$ have negative real parts if $a > r$. In other words, if $a > r$, then each eigenvalue is either real and negative, or complex with a negative real part. Check the veracity of this statement by finding the eigenvalues (use MAPLE) in the following two cases:

$$\textbf{(i)}\ a = 0.61,\ r = 0.6,\ m = 0.5,\ p = 0.4,\ v = 0.8,\ k = 1\,,$$

$$\textbf{(ii)}\ a = 0.59,\ r = 0.6,\ m = 0.5,\ p = 0.4,\ v = 0.8,\ k = 1\,.$$

c. Given the statement about the eigenvalues in part **b.**, how do you expect a solution in the full system to behave for the case $a > r$ if it starts near either E_1 or E_2?

5. The case $a = r$ corresponds to a special type of bifurcation known as a *Hopf bifurcation*. The behavior of solutions on either side of a Hopf bifurcation can be very interesting and complicated. Show that if $a = r$ is assumed, then the characteristic polynomial for E_1 factors into

$$p(\lambda) = (\lambda + m)\left(\lambda^2 + 2vr\left(1 + \frac{pk}{m}\right)\right).$$

Conclude that E_1 has one negative real eigenvalue and a pair of pure imaginary eigenvalues. Where have we seen this type of linear system before?

6. Use MAPLE and some of the results from the previous questions to investigate the solutions to the full system for the parameter values in the following cases:

$$\textbf{(i)}\ a = 0.26,\ r = 0.25,\ m = 0.2,\ p = 0.1,\ v = 0.5,\ k = 1\,,$$

$$\textbf{(ii)}\ a = 0.1,\ r = 0.25,\ m = 0.2,\ p = 0.1,\ v = 0.5,\ k = 1\,,$$

$$\textbf{(iii)}\ a = 0.26,\ r = 0.25,\ m = 0.2,\ p = 0.01,\ v = 0.03,\ k = 1\,,$$

$$\textbf{(iv)}\ a = 0.1,\ r = 0.25,\ m = 0.2,\ p = 0.01,\ v = 0.03,\ k = 1\,.$$

In each case, be sure to investigate the behavior of solutions for a variety of initial conditions. You may need to increase the length of time plotted to obtain valid conclusions, although sometimes MAPLE will not allow you to do this. Interpret your findings in terms of the economic model. Feel free to turn in any particularly revealing plots that you generate.

Reference: Lenka Pribylová, Bifurcation Routes to Chaos in an Extended Van Der Pol's Equation Applied to Economic Models, *Electronic Journal of Differential Equations*, Vol. 2009 (2009), No. 53, 1–21.