2. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-4t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-3t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 6\frac{dy_p}{dt} + 8y_p &= 9ke^{-3t} - 18ke^{-3t} + 8ke^{-3t} \\ &= -ke^{-3t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -2$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t} - 2e^{-3t}.$$

5. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= 4ke^{-2t} - 8ke^{-2t} + 13ke^{-2t} \\ &= 9ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t - \tfrac{1}{3} e^{-2t}.$$

6. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 7s + 10,$$

so the eigenvalues are $s = -2$ and $s = -5$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-5t}.$$

To find a particular solution of the forced equation, a reasonable looking guess is $y_p(t) = ke^{-2t}$. However, this guess is a solution of the homogeneous equation, so it is doomed to fail. We make the standard second guess of $y_p(t) = kte^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 7\frac{dy_p}{dt} + 10y_p &= (-4ke^{-2t} + 4kte^{-2t}) + 7(ke^{-2t} - 2kte^{-2t}) + 10kte^{-2t} \\ &= 3ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \tfrac{1}{3}te^{-2t}.$$

11. This is the same equation as Exercise 5. The general solution is

$$y(t) = k_1 e^{-2t}\cos 3t + k_2 e^{-2t}\sin 3t - \tfrac{1}{3}e^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t}\cos 3t - 3k_1 e^{-2t}\sin 3t - 2k_2 e^{-2t}\sin 3t + 3k_2 e^{-2t}\cos 3t + \tfrac{2}{3}e^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{1}{3} = 0 \\ -2k_1 + 3k_2 + \frac{2}{3} = 0. \end{cases}$$

Solving, we have $k_1 = 1/3$ and $k_2 = 0$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{3}e^{-2t}\cos 3t - \tfrac{1}{3}e^{-2t}.$$

12. This is the same equation as Exercise 6. The general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \tfrac{1}{3} t e^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t} + \tfrac{1}{3} e^{-2t} - \tfrac{2}{3} t e^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_1 - 5k_2 + \frac{1}{3} = 0. \end{cases}$$

Solving, we have $k_1 = -1/9$ and $k_2 = 1/9$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{9} e^{-2t} + \tfrac{1}{9} e^{-5t} + \tfrac{1}{3} t e^{-2t}.$$

31. **(a)** The general solution for the homogeneous equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

Suppose $y_p(t) = at^2 + bt + c$. Substituting $y_p(t)$ into the differential equation, we get

$$\frac{d^2 y_p}{dt^2} + 4y_p = -3t^2 + 2t + 3$$

$$2a + 4(at^2 + bt + c) = -3t^2 + 2t + 3$$

$$4at^2 + 4bt + (2a + 4c) = -3t^2 + 2t + 3.$$

Therefore, $y_p(t)$ is a solution if and only if

$$\begin{cases} 4a = -3 \\ 4b = 2 \\ 2a + 4c = 3. \end{cases}$$

Therefore, $a = -3/4$, $b = 1/2$, and $c = 9/8$. The general solution is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t - \tfrac{3}{4} t^2 + \tfrac{1}{2} t + \tfrac{9}{8}.$$

(b) To solve the initial-value problem, we use the initial conditions $y(0) = 2$ and $y'(0) = 0$ along with the general solution to form the simultaneous equations

$$\begin{cases} k_1 + \frac{9}{8} = 2 \\ 2k_2 + \frac{1}{2} = 0. \end{cases}$$

Therefore, $k_1 = 7/8$ and $k_2 = -1/4$. The solution is

$$y(t) = \tfrac{7}{8} \cos 2t - \tfrac{1}{4} \sin 2t - \tfrac{3}{4} t^2 + \tfrac{1}{2} t + \tfrac{9}{8}.$$

34. **(a)** To find a particular solution of the forced equation, we guess

$$y_p(t) = at^2 + bt + c.$$

Substituting this guess into the equation yields

$$(2a) + 3(2at + b) + 2(at^2 + bt + c) = t^2,$$

which can be rewritten as

$$(2a)t^2 + (6a + 2b)t + (2a + 3b + 2c) = t^2.$$

Equating coefficients, we have

$$\begin{cases} 2a = 1 \\ 6a + 2b = 0 \\ 2a + 3b + 2c = 0, \end{cases}$$

which gives $a = 1/2, b = -3/2$ and $c = 7/4$. So

$$y_p(t) = \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

To find the general solution of the unforced equation, we note that the characteristic polynomial

$$s^2 + 3s + 2$$

has roots $s = -2$ and $s = -1$, so the general solution for the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

(b) Note that

$$y'(t) = -2k_1 e^{-2t} - k_2 e^{-t} + t - \tfrac{3}{2}.$$

To satisfy the desired initial conditions, we compute

$$y(0) = k_1 + k_2 + \tfrac{7}{4}$$

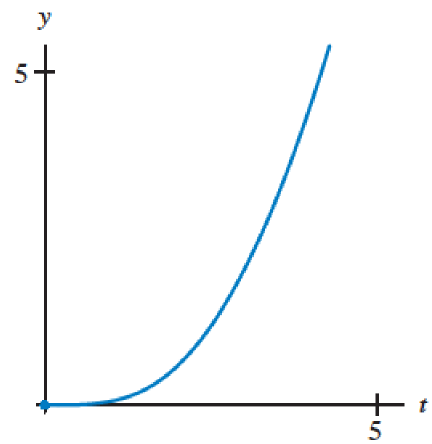
and

$$y'(0) = -2k_1 - k_2 - \tfrac{3}{2}.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 0$, we have $k_1 = 1/4$ and $k_2 = -2$. So the desired solution is

$$y(t) = \tfrac{1}{4}e^{-2t} - 2e^{-t} + \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

(c) The solution $\frac{1}{4}e^{-2t} - 2e^{-t}$ of the unforced equation tends to zero quickly, so the solution of the original equation tends to infinity at a rate that is determined by the quadratic $t^2/2 - 3t/2 + 7/4$. This rate is essentially the same as that of t^2.



1. The linearizations of systems (i) and (iii) are both

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -y,$$

so these two systems have the same "local picture" near $(0, 0)$. This system has eigenvalues 2 and -1; hence, $(0, 0)$ is a saddle for these systems. System (ii) has linearization

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = y,$$

which has eigenvalues 2 and 1, hence, $(0, 0)$ is a source for this system.

4. **(a)** The equilibrium points occur where the vector field is zero, that is, at solutions of

$$\begin{cases} -x = 0 \\ -4x^3 + y = 0. \end{cases}$$

So, $x = y = 0$ is the only equilibrium point.

(b) The Jacobian matrix of this system is

$$\begin{pmatrix} -1 & 0 \\ -12x^2 & 1 \end{pmatrix},$$

which at $(0, 0)$ is equal to

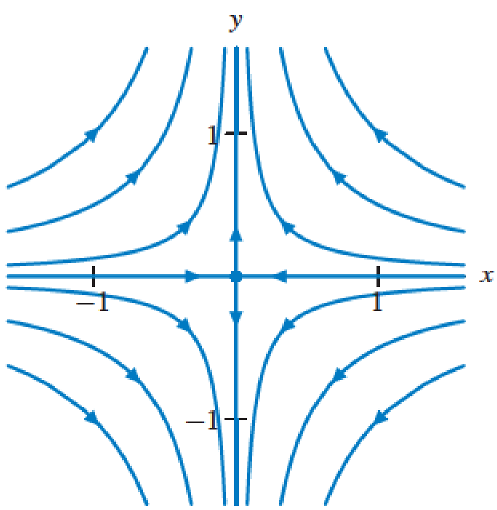
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the linearized system at $(0, 0)$ is

$$\frac{dx}{dt} = -x$$
$$\frac{dy}{dt} = y$$

(we could also see this by "dropping the higher order terms").

(c) The eigenvalues of the linearized system at the origin are -1 and 1, so the origin is a saddle. The linearized system decouples, so solutions approach the origin along the x-axis and tend away form the origin along the y-axis.



5. **(a)** Using separation of variables (or simple guessing), we have $x(t) = x_0 e^{-t}$.

(b) Using the result in part (a), we can rewrite the equation for dy/dt as

$$\frac{dy}{dt} = y - 4x_0^3 e^{-3t}.$$

This first-order equation is a nonhomogeneous linear equation.

The general solution of its associated homogeneous equation is ke^t. To find a particular solution to the nonhomogeneous equation, we rewrite it as

$$\frac{dy}{dt} - y = -4x_0^3 e^{-3t},$$

and we guess a solution of the form $y_p = \alpha e^{-3t}$. Substituting this guess into the left-hand side of the equation yields

$$\frac{dy_p}{dt} - y_p = -4\alpha e^{-3t}.$$

Therefore, y_p is a solution if $\alpha = x_0^3$. The general solution of the original equation is

$$y(t) = x_0^3 e^{-3t} + ke^t.$$

To express this result in terms of the initial condition $y(0) = y_0$, we evaluate at $t = 0$ and note that $k = y_0 - x_0^3$. We conclude that

$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

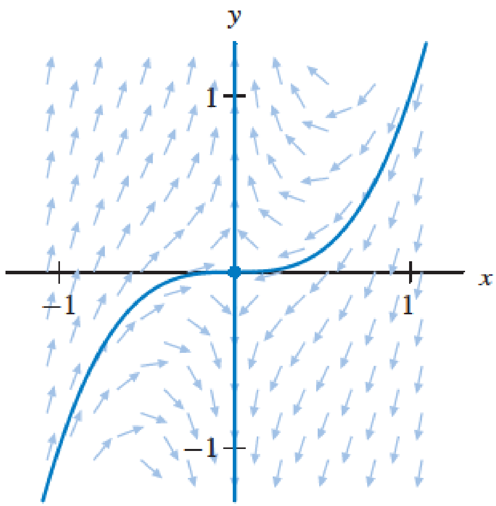
(c) The general solution of the system is

$$\begin{aligned} x(t) &= x_0 e^{-t} \\ y(t) &= x_0^3 e^{-3t} + (y_0 - x_0^3)e^t. \end{aligned}$$

(d) For all solutions, $x(t) \to 0$ as $t \to \infty$. For a solution to tend to the origin as $t \to \infty$, we must have $y(t) \to 0$, and this can happen only if $y_0 - x_0^3 = 0$.

(e) Since $x = x_0 e^{-t}$, we see that a solution will tend toward the origin as $t \to -\infty$ only if $x_0 = 0$. In that case, $y(t) = y_0 e^t$, and $y(t) \to 0$ as $t \to -\infty$.

(f)



(g) Solutions tend away from the origin along the y-axis in both systems. In the nonlinear system, solutions approach the origin along the curve $y = x^3$ which is tangent to the x-axis. For the linearized system, solutions tend to the origin along the x-axis. Near the origin, the phase portraits are almost the same.

9. **(a)** The equilibrium points are $(0, 0)$, $(0, 25)$, $(100, 0)$ and $(75, 12.5)$. We classify these equilibrium points by computing the Jacobian matrix, which is

$$\begin{pmatrix} 100 - 2x - 2y & -2x \\ -y & 150 - x - 12y \end{pmatrix},$$

and evaluating it at each of the equilibrium points. At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 150 \end{pmatrix},$$

and the eigenvalues are 100 and 150. So this point is a source. At $(0, 25)$, the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ -25 & -150 \end{pmatrix},$$

and the eigenvalues are 50 and -150. Hence, this point is a saddle. At $(100, 0)$, the Jacobian matrix is

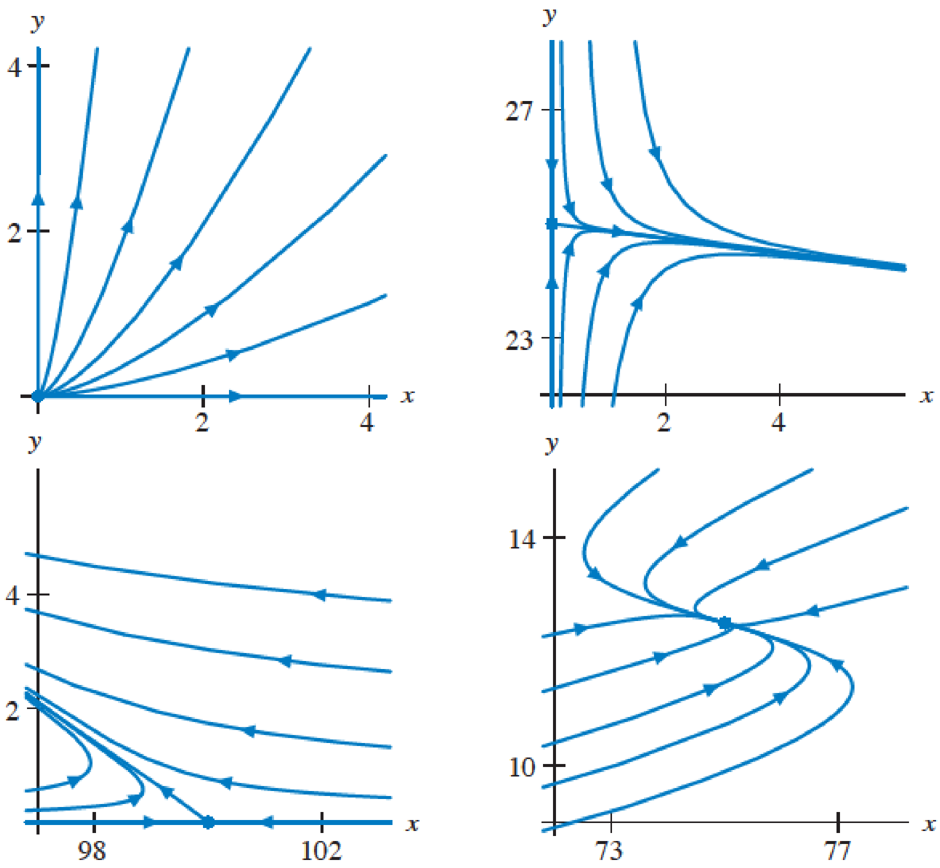
$$\begin{pmatrix} -100 & -200 \\ 0 & 50 \end{pmatrix},$$

and the eigenvalues are -100 and 50. Therefore, this point is a saddle. Finally, at $(75, 12.5)$, the Jacobian matrix is

$$\begin{pmatrix} -75 & -150 \\ -12.5 & -75 \end{pmatrix},$$

and the eigenvalues are approximately -32 and -118. So this point is a sink.

(b)



14. **(a)** The equilibrium points are $(0, 0)$, $(1, 1)$ and $(2, 0)$. We classify these points by calculating the Jacobian matrix, which is,

$$\begin{pmatrix} 2 - 2x - y & -x \\ -2xy & 2y - x^2 \end{pmatrix},$$

and evaluating it at the points. At $(0, 0)$, the Jacobian is

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

which has eigenvalues 2 and 0. An eigenvector for the eigenvalue 2 is $(1, 0)$, so solutions move away from the origin parallel to the x-axis. On the line $x = 0$, we have $dy/dt = y^2$ so solutions move upwards when $y \neq 0$. Hence, $(0, 0)$ is a node. However, solutions near the origin in the first quadrant move away from the origin as t increases. At $(1, 1)$, the Jacobian is

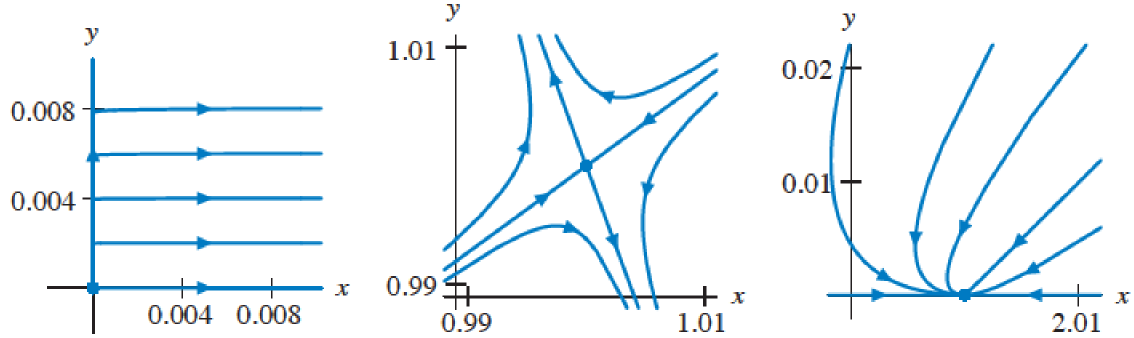
$$\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{3}$. So $(1, 1)$ is a saddle. At $(2, 0)$, the Jacobian is

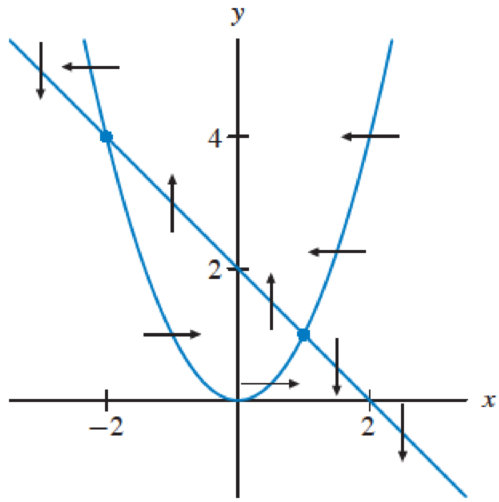
$$\begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix},$$

which has eigenvalues -2 and -4. Thus, $(2, 0)$ is a sink.

(b)

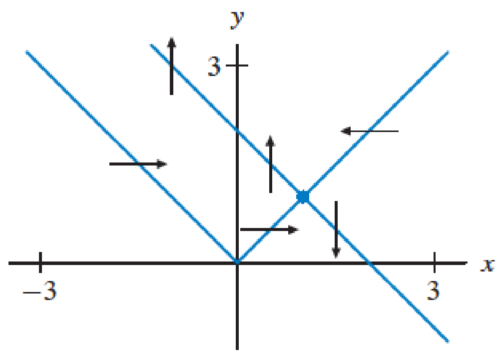


1. For x- and y-nullclines, $dx/dt = 0$, and $dy/dt = 0$ respectively. Then, we obtain $y = -x + 2$ for the x-nullcline and $y = x^2$ for the y-nullcline. To find intersections, we set $-x + 2 = x^2$, or $(x+2)(x-1) = 0$. Solving this for x yields $x = 1, -2$. For $x = 1$, $y = 1$, and for $x = -2$, $y = 4$. So the equilibrium points are $(1, 1)$ and $(-2, 4)$.



The solution for (a) is in the left-down region, and therefore, it eventually enters the region where $y < -x + 2$ and $y < x^2$. Once the solution enters this region, it stays there because the vector field on the boundaries never points out. Solutions for (b) and (c) start in this same region. Hence, all three solutions will go down and to the right without bound.

2. For x- and y-nullclines, $dx/dt = 0$, and $dy/dt = 0$ respectively. So, we have $y = -x + 2$ for the x-nullcline and $y = |x|$ for the y-nullcline. To find intersections, we set $-x + 2 = |x|$. Solving this for x yields $x = 1$, so the only equilibrium point is $(x, y) = (1, 1)$.



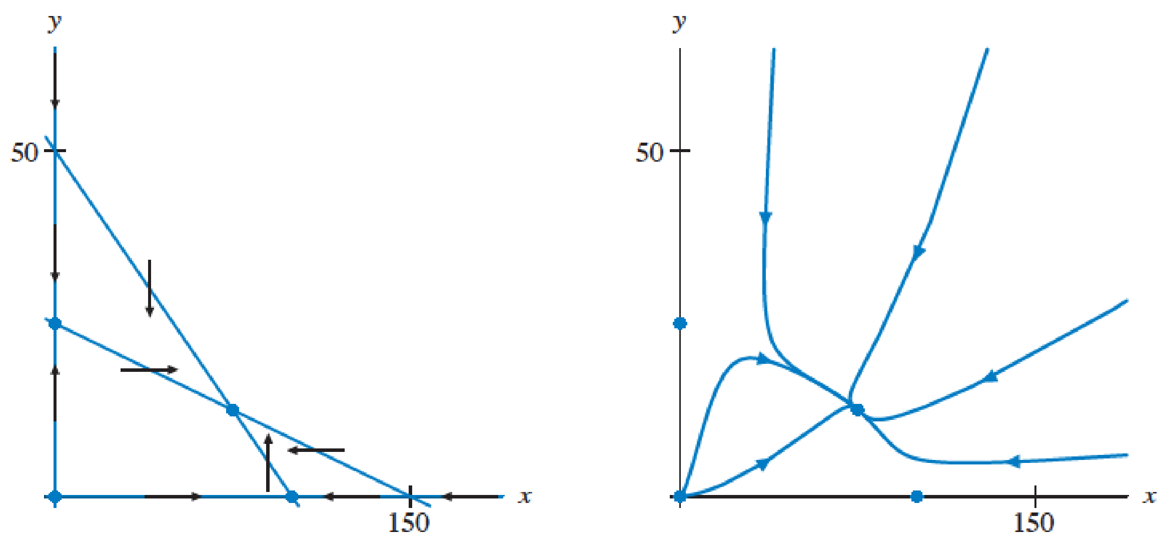
The solution for (a) begins on the y-nullcline, heads into the right-up region, eventually crosses the x-nullcline, and then tends to infinity in the left-up region.

The solution for (b) starts in the left-down position, crosses the x-nullcline, then tends to infinity in the right-down region.

The solution corresponding to (c) starts on the y-nullcline, immediately enters the left-up region, and then tends to infinity in this region.

7. **(a)** The x-nullcline consists of the two lines $x = 0$ and $y = -x/2 + 50$. The y-nullcline consists of the two lines $y = 0$ and $y = -x/6 + 25$.

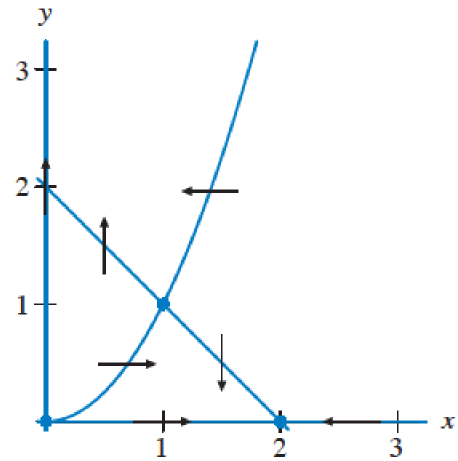
(b)

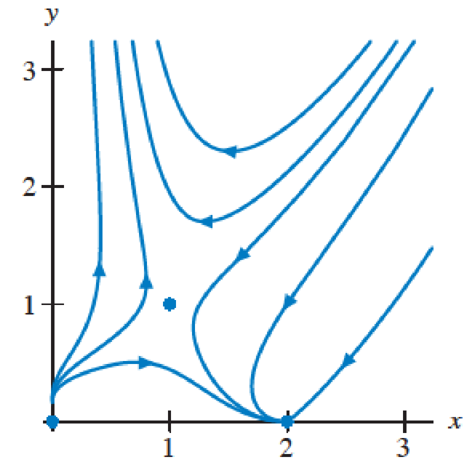


(c) All solutions off the axes tend toward the sink at $(75, 25/2)$. On the x-axis, solutions tend to the saddle at $(100, 0)$. On the y-axis, solutions tend to the saddle at $(0, 25)$.

12. **(a)** The x-nullcline is given by the lines $x = 0$ and $y = -x + 2$. The y-nullcline is given by the line $y = 0$ and the curve $y = x^2$.

(b)





(c) There is a saddle point at $(1, 1)$. Two solutions leave this equilibrium point, one tending to infinity, the other to the equilibrium point at $(2, 0)$. Two solutions tend toward the saddle point, one coming from the origin, one from infinity. To the "right" of and "below" the incoming solution curve, all solutions tend to the equilibrium point at $(2, 0)$; to the "left" all solutions tend to ∞.

1. **(a)** We compute that

$$\frac{\partial H}{\partial x} = x - x^3$$

and so

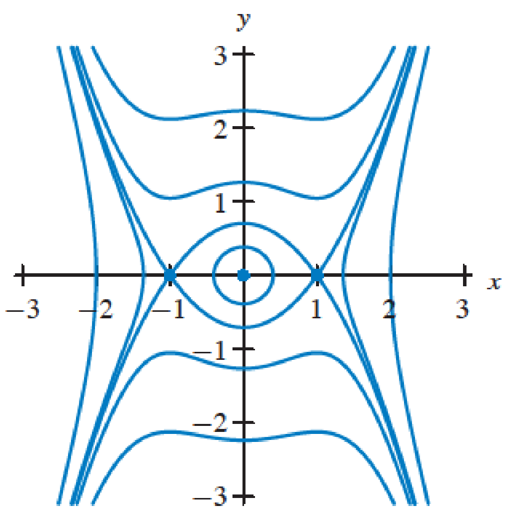
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

Also,

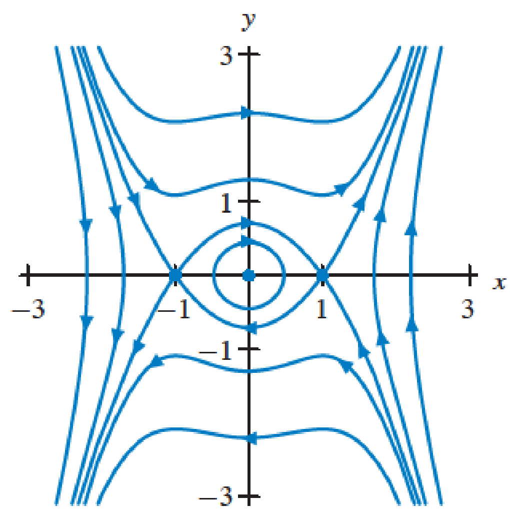
$$\frac{\partial H}{\partial y} = y = \frac{dx}{dt}.$$

Hence, this is a Hamiltonian system with Hamiltonian function H.

(b) Note that $(0, 0)$ is a local minimum and $(\pm 1, 0)$ are saddle points.



(c) The equilibrium point $(0, 0)$ is a center and $(\pm 1, 0)$ are saddles. The saddles are connected by separatrix solutions.



2. **(a)** If $H(x, y) = \sin(xy)$, then

$$\frac{\partial H}{\partial x} = y\cos(xy)$$

and so

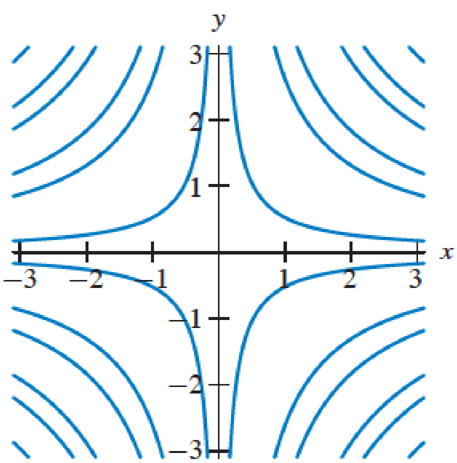
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

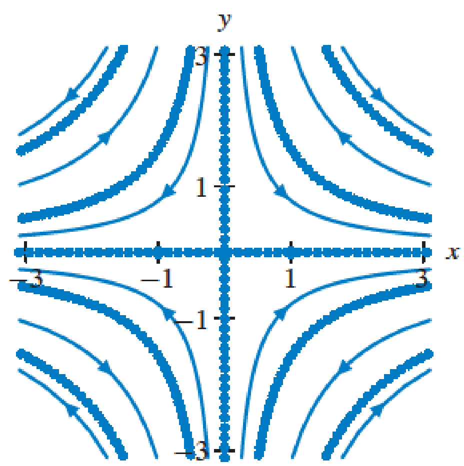
Similarly,

$$\frac{\partial H}{\partial y} = x\cos(xy) = \frac{dx}{dt}.$$

(b) Note that the level sets of H are the same curves as those of the level sets of xy.

(c) Note that there are many curves of equilibrium points for this system: besides the origin, whenever $xy = n\pi + \pi/2$, the vector field vanishes.





9. We know that the equilibrium points of a Hamiltonian system cannot be sources or sinks. Phase portrait (b) has a spiral source, so it is not Hamiltonian. Phase portrait (c) has a sink and a source, so it is not Hamiltonian. Phase portraits (a) and (d) might come from Hamiltonian systems. (Try to imagine a function which has the solution curves as level sets.)

10. First note that

$$\frac{\partial(\sin x \cos y)}{\partial x} = \cos x \cos y = -\frac{\partial(2x - \cos x \sin y)}{\partial y}.$$

Hence, the system is Hamiltonian. Integrating dx/dt with respect to y yields

$$H(x, y) = \sin x \sin y + c(x).$$

If we differentiate $H(x, y)$ with respect to x, we get

$$\cos x \sin y + c'(x),$$

which we want to be the negative of $dy/dt = 2x - \cos x \sin y$. Hence $c'(x) = -2x$, and we pick the antiderivative $c(x) = -x^2$. A Hamiltonian function is

$$H(x, y) = -x^2 + \sin x \sin y.$$

11. First note that

$$\frac{\partial(x - 3y^2)}{\partial x} = 1 = -\frac{\partial(-y)}{\partial y}.$$

Hence, the system is Hamiltonian. Integrating dx/dt with respect to y yields

$$H(x, y) = xy - y^3 + c(x).$$

If we differentiate $H(x, y)$ with respect to x, we get

$$y + c'(x),$$

which we want to be the negative of $dy/dt = -y$. Hence $c'(x) = 0$, and we pick the antiderivative $c(x) = 0$. A Hamiltonian function is

$$H(x, y) = xy - y^3.$$

12. First we check to see if the partial derivative with respect to x of the first component of the vector field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial 1}{\partial x} = 0$$

while

$$-\frac{\partial y}{\partial y} = -1.$$

Since these are not equal, the system is not Hamiltonian.