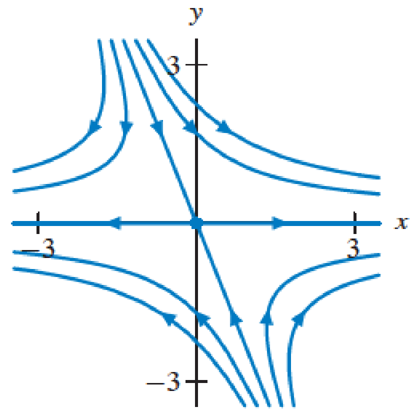
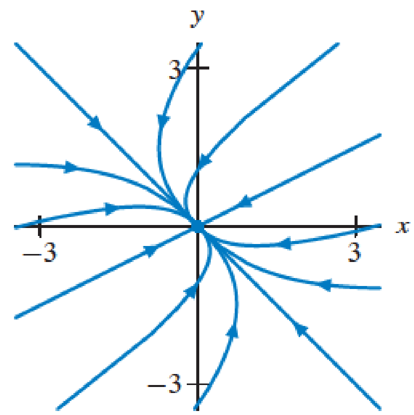
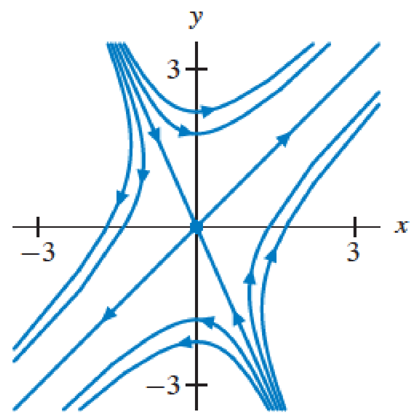
**1.** As we computed in Exercise 1 of Section 3.2, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. The eigenvectors $(x_1, y_1)$ for the eigenvalue $\lambda_1 = -2$ satisfy $5x_1 = -2y_1$, and the eigenvectors $(x_2, y_2)$ for $\lambda_2 = 3$ satisfy the equation $y_2 = 0$. The equilibrium point at the origin is a saddle.




**2.** As we computed in Exercise 2 of Section 3.2, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -5$. The eigenvectors $(x_1, y_1)$ for the eigenvalue $\lambda_1 = -2$ satisfy $y_1 = -x_1$, and the eigenvectors $(x_2, y_2)$ for $\lambda_2 = -5$ satisfy $x_2 = 2y_2$. The equilibrium point at the origin is a sink.



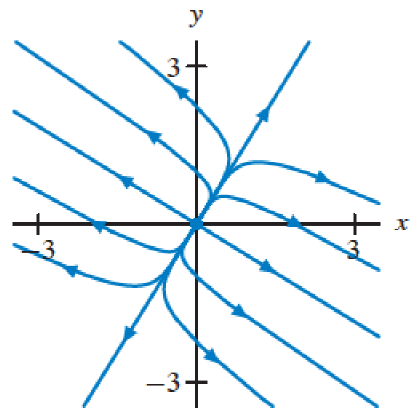

**4.** As we computed in Exercise 6 of Section 3.2, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$. The eigenvectors $(x_1, y_1)$ for the eigenvalue $\lambda_1 = -4$ satisfy $9x_1 = -4y_1$, and the eigenvectors $(x_2, y_2)$ for $\lambda_2 = 9$ satisfy the equation $y_2 = x_2$. The equilibrium point at the origin is a saddle.



**6.** As we computed in Exercise 8 of Section 3.2, the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

The eigenvectors $(x_1, y_1)$ for the eigenvalue $\lambda_1$ satisfy $y_1 = (1-\sqrt{5})x_1/2$, and the eigenvectors $(x_2, y_2)$ for the eigenvalue $\lambda_2$ satisfy $y_2 = (1+\sqrt{5})x_2/2$. The equilibrium point at the origin is a source.



**20.** **(a)** The characteristic equation is

$$(2-\lambda)(-2-\lambda) - 12 = \lambda^2 - 16 = 0,$$

so the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 4$. Therefore, the equilibrium point at the origin is a saddle.

**(b)** To find all the straight-line solutions, we must calculate the eigenvectors. For the eigenvalue $\lambda_1 = -4$, we have the simultaneous equations

$$\begin{cases} 2x_1 + 6y_1 = -4x_1 \\ 2x_1 - 2y_1 = -4y_1, \end{cases}$$

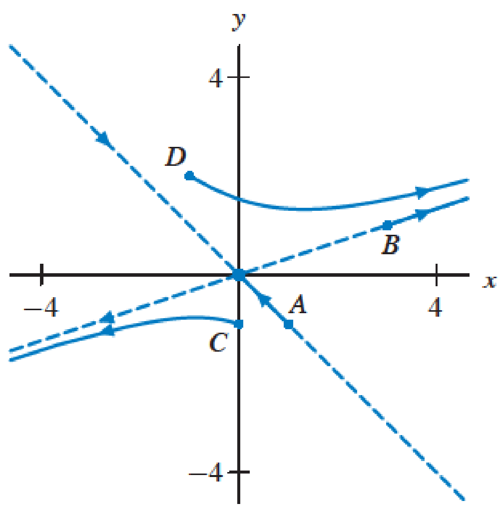
and we obtain $y_1 = -x_1$. In other words, all vectors on the line $y_1 = -x_1$ are eigenvectors for $\lambda_1$. Therefore, any solution of the form $e^{-4t}(x_1, -x_1)$ for any $x_1$ is a straight-line solution corresponding to the eigenvalue $\lambda_1 = -4$.

To calculate the eigenvectors associated to the eigenvalue $\lambda_2 = 4$, we must solve the equations

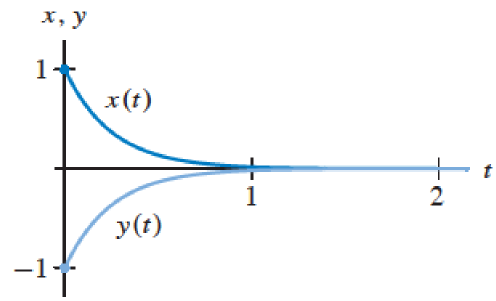
$$\begin{cases} 2x_2 + 6y_2 = 4x_2 \\ 2x_2 - 2y_2 = 4y_2, \end{cases}$$

and we obtain $x_2 = 3y_2$. Therefore, any solution of the form $e^{4t}(3y_2, y_2)$ for any $y_2$ is a straight-line solution corresponding to the eigenvalue $\lambda_2 = 4$.

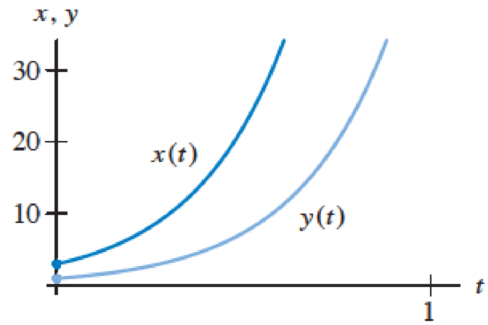
**(c)** In the phase plane, the only solution curves that approach the origin are those whose initial conditions lie on the line $y = -x$. All other solution curves eventually approach those that correspond to the line $x = 3y$.



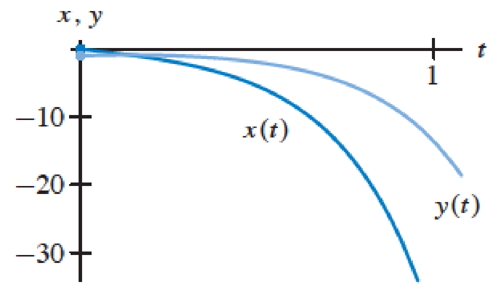
The initial condition $A = (1, -1)$ lies on the line $y = -x$. Therefore, it corresponds to a straight-line solution. In fact, the formula for its solution is $e^{-4t}(1, -1)$.



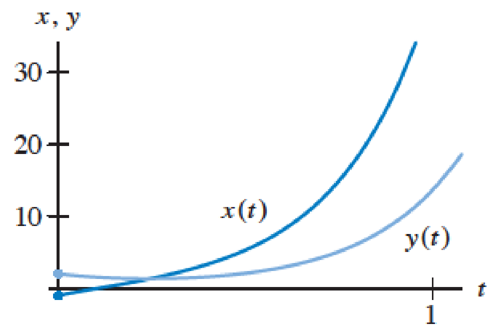

The initial condition $B = (3, 1)$ lies on the line $x = 3y$. Therefore, it corresponds to a straight-line solution, and the formula is $e^{4t}(3, 1)$.




The solution curve that corresponds to the initial condition $C = (0, -1)$ enters the third quadrant and eventually approaches line $x = 3y$. From the phase plane, we see that $x(t)$ is decreasing for all $t > 0$. We also see that $y(t)$ increases initially, reaches a negative maximum value, and then decreases in an exponential fashion. Since the solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line where $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its maximum value.




The solution curve that corresponds to the initial condition $D = (-1, 2)$ moves from the second quadrant into the first quadrant and eventually approaches the line $x = 3y$. From the phase plane, we see that $x(t)$ is increasing for all $t > 0$. We also see that $y(t)$ decreases initially, reaches a positive minimum value, and then increases in an exponential fashion. Since this solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line for which $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its minimum value.



**21.** **(a)** The second-order equation is

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 0.$$

Introducing $v = dy/dt$, we obtain the system

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -6y - 7v.\end{aligned}$$

**(b)** The characteristic polynomial is

$$\lambda^2 + 7\lambda + 6,$$

which factors into $(\lambda + 6)(\lambda + 1)$.

**(c)** From the characteristic polynomial, we obtain the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -1$.

**(d)** To compute the eigenvectors associated to $\lambda_1 = -6$, we solve the simultaneous equations

$$\begin{cases} v = -6y \\ -6y - 7v = -6v. \end{cases}$$

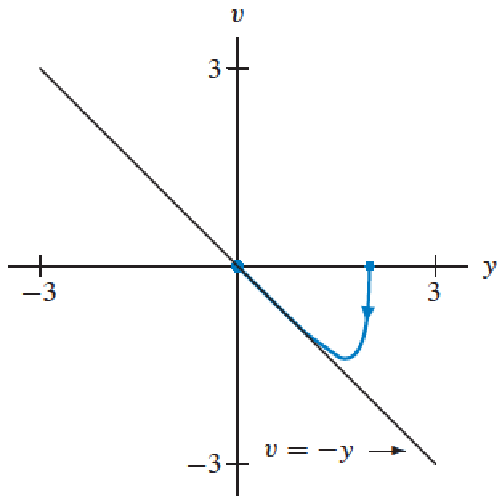
Therefore, any vector on the line $v = -6y$ is an eigenvector associated to the eigenvalue $\lambda_1$.

To compute the eigenvectors associated to $\lambda_2 = -1$, we must solve the simultaneous equations

$$\begin{cases} v = -y \\ -6y - 7v = -v. \end{cases}$$

Therefore, any vector on the line $y = -v$ is an eigenvector associated to the eigenvalue $\lambda_2$.

Since both eigenvalues are real and negative, we know that origin is a sink, and the solution curve corresponding to the initial condition $(y(0), v(0)) = (2, 0)$ tends toward the origin tangent to the line $y = -v$ in the $yv$-plane.




From the phase portrait, we see that the solution curve remains in the fourth quadrant for all $t > 0$. Consequently, it does not cross the line $y = 0$, and the mass cannot cross the equilibrium position. The solution approaches the origin at the rate that is determined by the eigenvalue $\lambda_2 = -1$. In other words, it approaches the origin at the rate of $e^{-t}$.

**22.** The differential equation is

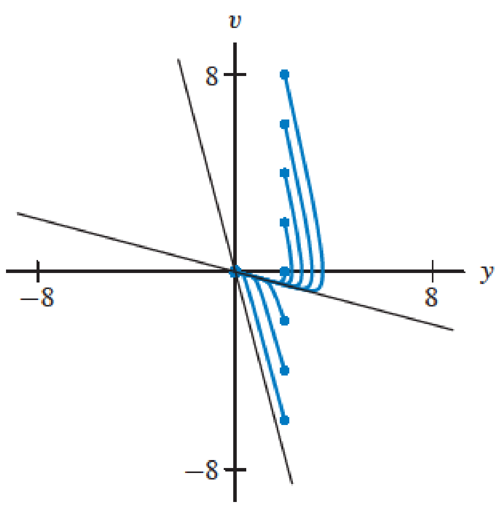
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0,$$

which corresponds to the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y - 4v.$$

The characteristic polynomial is $\lambda^2 + 4\lambda + 1$, and consequently the eigenvalues are $\lambda = -2 \pm \sqrt{3}$.

The eigenvectors for $\lambda = -2 + \sqrt{3}$ satisfy $v = (-2 + \sqrt{3})y$, and the eigenvectors for $\lambda = -2 - \sqrt{3}$ satisfy $v = (-2 - \sqrt{3})y$. Looking at the phase plane, the line $y = 2$ crosses each line of eigenvectors once. The line of eigenvectors corresponding to $\lambda = -2 - \sqrt{3}$ is crossed at $v = -4 - 2\sqrt{3}$ while the line of eigenvectors corresponding to $\lambda = -2 + \sqrt{3}$ is crossed at $v = -4 + 2\sqrt{3}$.




The solutions with $y = 2$, $v < -4 - 2\sqrt{3}$ all cross into the left-half ($y < 0$ half) of the phase plane. In other words, if the initial velocity is sufficiently negative, then $y$ overshoots $y = 0$. For $v \geq -4 - 2\sqrt{3}$, $y(t)$ remains positive for all $t$. Solutions tending toward the origin most quickly are those on the line of eigenvectors corresponding to the more negative eigenvalue, so the solution that reaches 0.1 quickest is the one whose initial velocity is $v = -4 - 2\sqrt{3}$.

**1.** Using Euler's formula, we can write the complex-valued solution $\mathbf{Y}_c(t)$ as

$$\begin{aligned}
\mathbf{Y}_c(t) &= e^{(1+3i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t e^{3it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t(\cos 3t + i \sin 3t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + i e^t \begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + k_2 e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

**3.** **(a)** The characteristic equation is

$$(-\lambda)^2 + 4 = \lambda^2 + 4 = 0,$$

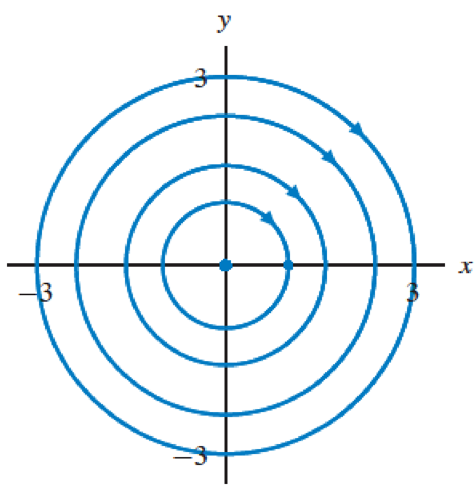
and the eigenvalues are $\lambda = \pm 2i$.

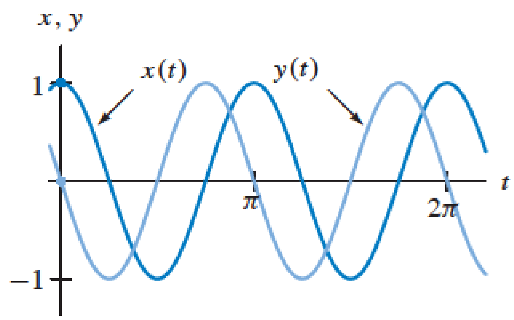
**(b)** Since the real part of the eigenvalues are 0, the origin is a center.

**(c)** Since $\lambda = \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

**(d)** At $(1, 0)$, the tangent vector is $(-2, 0)$. Therefore, the direction of oscillation is clockwise.

**(e)** According to the phase plane, $x(t)$ and $y(t)$ are periodic with period $\pi$. At the initial condition $(1, 0)$, both $x(t)$ and $y(t)$ are initially decreasing.



**4.** **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda)+8=\lambda^2-8\lambda+20,$$

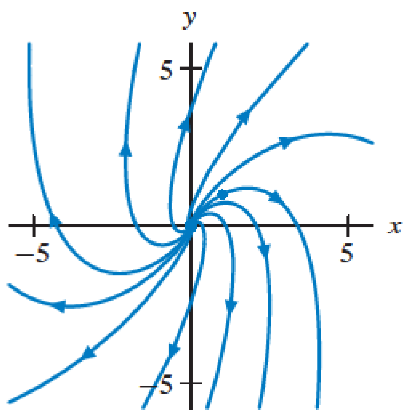
and the eigenvalues are $\lambda = 4 \pm 2i$.

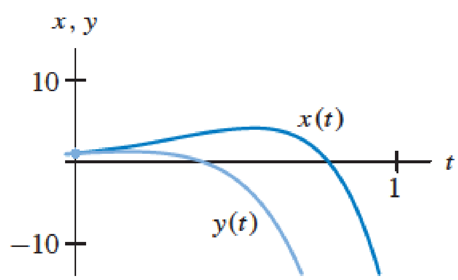
**(b)** Since the real part of the eigenvalues is positive, the origin is a spiral source.

**(c)** Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

**(d)** At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

**(e)** Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is $\pi$, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.



**6.** **(a)** The characteristic polynomial is

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4,$$

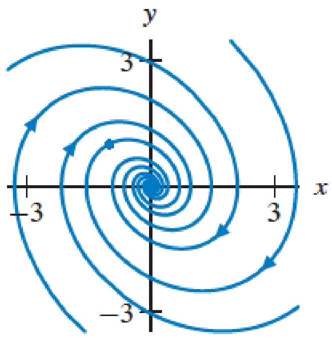
so the eigenvalues are $\lambda = (-1 \pm i\sqrt{15})/2$.

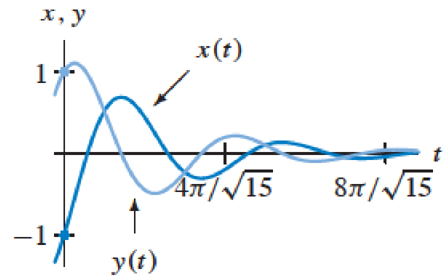
**(b)** The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

**(c)** The natural period is $2\pi/(\sqrt{15}/2) = 4\pi/\sqrt{15}$. The natural frequency is $\sqrt{15}/(4\pi)$.

**(d)** The vector field at $(1, 0)$ is $(0, -2)$. Hence, solution curves spiral about the origin in a clockwise fashion.

**(e)** From the phase plane, we see that both $x(t)$ and $y(t)$ are initially increasing. However, $y(t)$ quickly reaches a local maximum. Although both functions oscillate, each successive oscillation has a smaller amplitude.



**9.** **(a)** According to Exercise 3, $\lambda = \pm 2i$. The eigenvectors $(x, y)$ associated to eigenvalue $\lambda = 2i$ must satisfy the equation $2y = 2ix$, which is equivalent to $y = ix$. One such eigenvector is $(1, i)$, and thus we have the complex solution

$$\mathbf{Y}(t) = e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

Taking real and imaginary parts, we obtain the general solution

$$\mathbf{Y}(t) = k_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + k_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

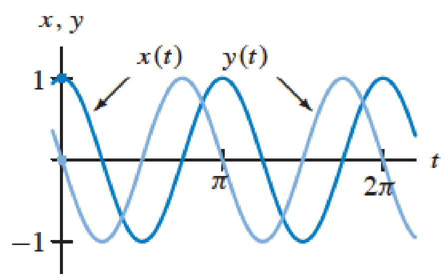
**(b)** From the initial condition, we obtain

$$k_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and therefore, $k_1 = 1$ and $k_2 = 0$. Consequently,the solution with the initial condition $(1, 0)$ is

$$\mathbf{Y}(t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}.$$

**(c)**



**10.** **(a)** According to Exercise 4, the eigenvalues are $\lambda = 4 \pm 2i$. The eigenvectors $(x, y)$ associated to the eigenvalue $4 + 2i$ must satisfy the equation $y = (1 + i)x$. Hence, using the eigenvector $(1, 1 + i)$, we obtain the complex-valued solution

$$\mathbf{Y}(t) = e^{(4+2i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + i e^{4t} \begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

From the real and imaginary parts of $\mathbf{Y}(t)$, we obtain the general solution

$$\mathbf{Y}(t) = k_1 e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

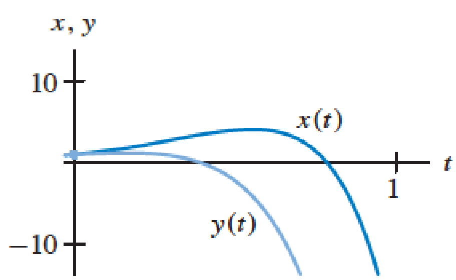
**(b)** Using the initial condition, we have

$$k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus $k_1 = 1$ and $k_2 = 0$. The desired solution is

$$\mathbf{Y}(t) = e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

**(c)**




**20.** If $\mathbf{AY}_0 = \lambda \mathbf{Y}_0$, then we can take complex conjugates of both sides to obtain $\overline{\mathbf{AY}_0} = \overline{\lambda \mathbf{Y}_0}$ (where the complex conjugate of a vector or matrix is the complex conjugate of its entries). But $\overline{\mathbf{AY}_0} = \overline{\mathbf{A}}\overline{\mathbf{Y}_0} = \mathbf{A}\overline{\mathbf{Y}_0}$ because $\mathbf{A}$ is real. Also, $\overline{\lambda \mathbf{Y}_0} = \overline{\lambda}\,\overline{\mathbf{Y}_0}$. Hence, $\mathbf{A}\overline{\mathbf{Y}_0} = \overline{\lambda}\overline{\mathbf{Y}_0}$. In other words, $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$ with eigenvector $\overline{\mathbf{Y}_0}$.

**1.** **(a)** The characteristic equation is

$$(-3-\lambda)^2 = 0,$$

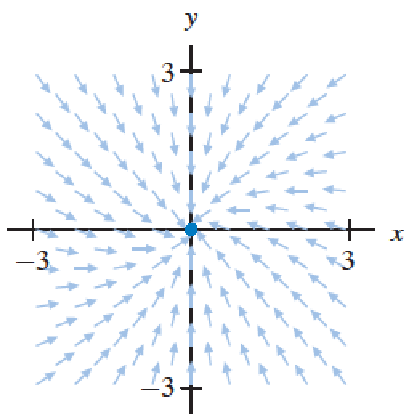
and the eigenvalue is $\lambda = -3$.

**(b)** To find an eigenvector, we solve the simultaneous equations

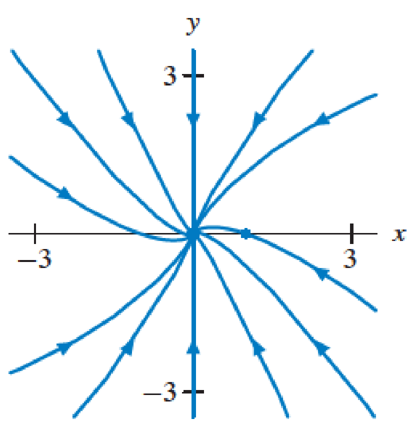
$$\begin{cases} -3x = -3x \\ x - 3y = -3y. \end{cases}$$

Then, $x = 0$, and one eigenvector is $(0, 1)$.

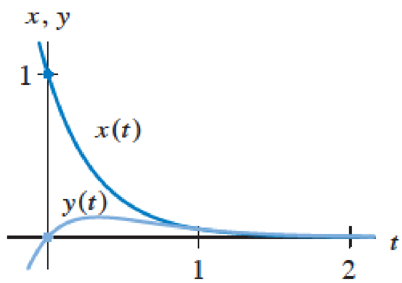
**(c)** Note the straight-line solutions along the $y$-axis.




**(d)** Since the eigenvalue is negative, any solution with an initial condition on the $y$-axis tends toward the origin as $t$ increases. According to the direction field, every solution tends to the origin as $t$ increases. The solutions with initial conditions in the half-plane $x > 0$ eventually approach the origin along the positive $y$-axis. Similarly, the solutions with initial conditions in the half-plane $x < 0$ eventually approach the origin along the negative $y$-axis.



**(e)** At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-3, 1)$. Therefore, $x(t)$ decreases initially and $y(t)$ increases initially. The solution eventually approaches the origin tangent to the positive $y$-axis. Therefore, $x(t)$ monotonically decreases to zero and $y(t)$ eventually decreases toward zero. Since the solution with the initial condition $\mathbf{Y}_0$ never crosses $y$-axis in the phase plane, the function $x(t) > 0$ for all $t$.



**2.** **(a)** The characteristic polynomial is

$$(2-\lambda)(4-\lambda)+1=\lambda^2-6\lambda+9=(\lambda-3)^2,$$

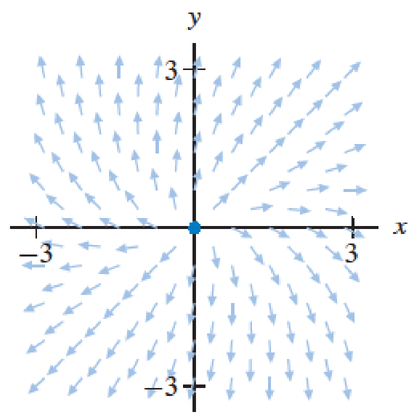
so there is only one eigenvalue, $\lambda = 3$.

**(b)** To find an eigenvector, we solve the equations

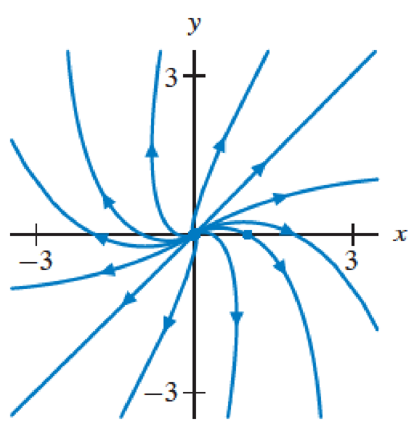
$$\begin{cases} 2x+y=3x \\ -x+4y=3y. \end{cases}$$

Both equations simplify to $y = x$, so $(1, 1)$ is one eigenvector.

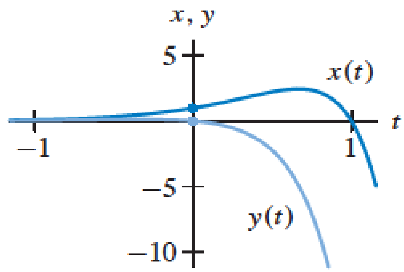
**(c)** Note the straight-line solutions along the line $y = x$.




**(d)** Since the sole eigenvalue is positive, all solutions except the equilibrium solution are unbounded as $t$ increases. As $t \to -\infty$, the solutions with initial conditions in the half-plane $y > x$ tend to the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, solutions with initial conditions in the half-plane $y < x$ tend to the origin tangent to the half-line $y = x$ with $y > 0$. Note the solution curve that goes through the initial condition $(1, 0)$.



**(e)** At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (2, -1)$. Hence, $x(t)$ is initially increasing, and $y(t)$ is initially decreasing.



**5.** **(a)** According to Exercise 1, there is one eigenvalue, $-3$, with eigenvectors of the form $(0, y_0)$, where $y_0 \neq 0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned}\mathbf{V}_1 &= \left[\begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ x_0 \end{pmatrix}.\end{aligned}$$

We obtain the general solution

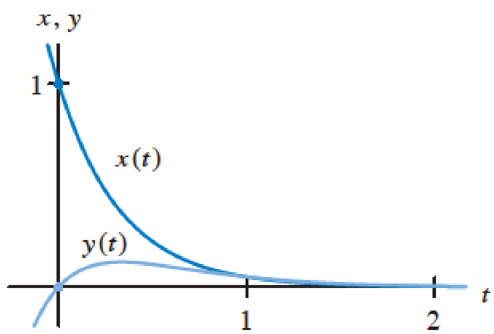
$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ x_0 \end{pmatrix}.$$

**(b)** The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $x(t) = e^{-3t}$ and $y(t) = te^{-3t}$.

**(c)** Compare the graphs of $x(t) = e^{-3t}$ and $y(t) = te^{-3t}$ with the sketches obtained in part (e) of Exercise 1.



**6.** **(a)** From Exercise 2, we know that there is only one eigenvalue, $\lambda = 3$, and the eigenvectors $(x_0, y_0)$ satisfy the equation $y_0 = x_0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned}\mathbf{V}_1 &= \left[\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0 \\ &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.\end{aligned}$$

We obtain the general solution

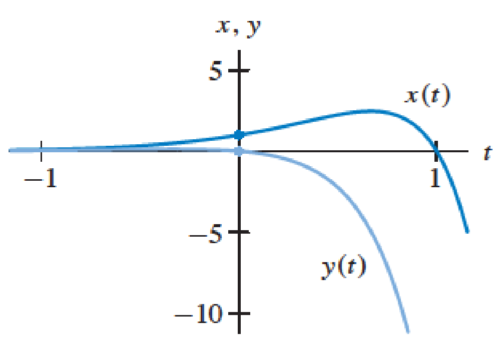
$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.$$

**(b)** The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{3t}(1 - t)$ and $y(t) = -te^{3t}$.

**(c)** Compare the graphs of $x(t) = e^{3t}(1-t)$ and $y(t) = -te^{3t}$ with the sketches obtained in part (e) of Exercise 2.




**12.** The characteristic polynomial of **A** is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

(see Section 3.2). A quadratic polynomial has only one root if and only if its discriminant is 0. In this case, the discriminant of $\det(\mathbf{A} - \lambda\mathbf{I})$ is $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})$.

**18.** **(a)** The characteristic equation is

$$(2 - \lambda)(6 - \lambda) - 12 = \lambda^2 - 8\lambda = 0.$$

Therefore, the eigenvalues are $\lambda = 0$ and $\lambda = 8$.

**(b)** To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2x_1 + 4y_1 = 0 \\ 3x_1 + 6y_1 = 0, \end{cases}$$

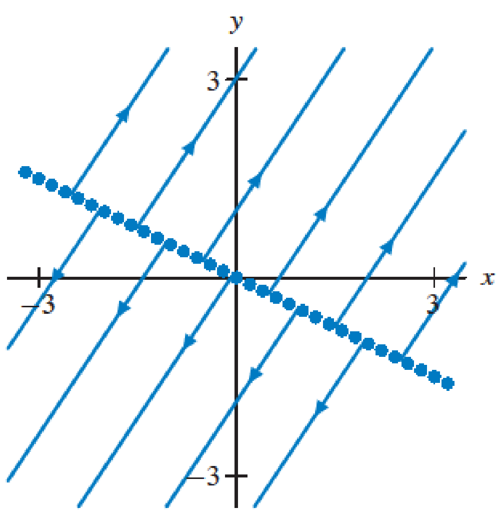
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $x_1 + 2y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 8$, we must solve $\mathbf{A}\mathbf{V}_2 = 8\mathbf{V}_2$. We get

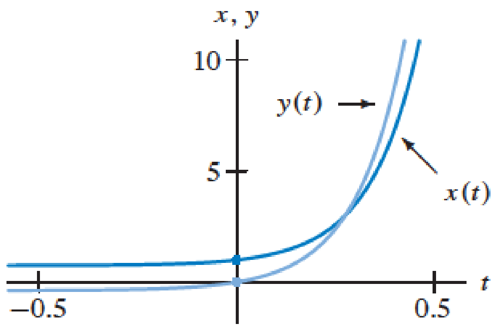
$$\begin{cases} 2x_2 + 4y_2 = 8x_2 \\ 3x_2 + 6y_2 = 8y_2, \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 8$ must satisfy $2y_2 = 3x_2$.

**(c)** The equation $x_1 + 2y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as $t$ increases.



**(d)** As $t$ increases, both $x(t)$ and $y(t)$ increase exponentially. As $t$ decreases, both $x$ and $y$ approach constants that are determined by the line of equilibrium points.




**(e)** To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (-2, 1)$, and $\mathbf{V}_2 = (2, 3)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

**(f)** To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

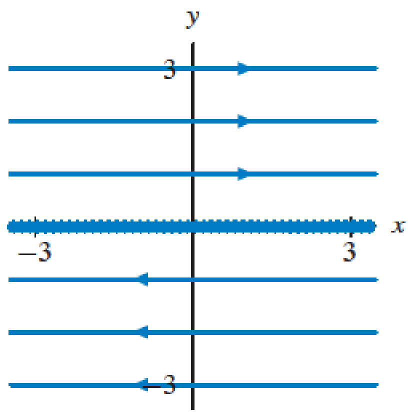
Therefore, $k_1 = -3/8$ and $k_2 = 1/8$. The particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}e^{8t} \\ -\frac{3}{8} + \frac{3}{8}e^{8t} \end{pmatrix}.$$

**20.** **(a)** The characteristic equation is $\lambda^2 - (a + d)\lambda + ad - bc = 0$. If 0 is an eigenvalue of $\mathbf{A}$, then 0 is a root of the characteristic polynomial. Thus, the constant term in the above equation must be 0—that is, $ad - bc = \det \mathbf{A} = 0$.

**(b)** If $\det \mathbf{A} = 0$, then the characteristic equation becomes $\lambda^2 - (a + d)\lambda = 0$, and this equation has 0 as a root. Therefore 0 is an eigenvalue of $\mathbf{A}$.

**21.** **(a)** The characteristic polynomial is $\lambda^2 = 0$, so $\lambda = 0$ is the sole eigenvalue. To sketch the phase portrait we note that $dy/dt = 0$, so $y(t)$ is always a constant function. Moreover, $dx/dt = 2y$, so $x(t)$ is increasing if $y > 0$, and it is decreasing if $y < 0$.




**(b)** This system is exactly the same as the one in part (a) except that the sign of $dx/dt$ has changed. Hence, the phase portrait is the identical except for the fact that the arrows point the other way.