6. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix} \mathbf{Y}$

7. $\mathbf{Y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix} \mathbf{Y}$

8. $\dfrac{dx}{dt} = -3x + 2\pi y$

$\dfrac{dy}{dt} = 4x - y$

14. **(a)** If $a = 0$, then $\det \mathbf{A} = ad - bc = bc$. Thus both b and c are nonzero if $\det \mathbf{A} \neq 0$.

(b) Equilibrium points (x_0, y_0) are solutions of the simultaneous system of linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a = 0$, the first equation reduces to $by_0 = 0$, and since $b \neq 0$, $y_0 = 0$. In this case, the second equation reduces to $cx_0 = 0$, so $x_0 = 0$ as well. Therefore, $(x_0, y_0) = (0, 0)$ is the only equilibrium point for the system.

24. **(a)** Substituting $\mathbf{Y}_1(t)$ in the left-hand side of the differential equation yields

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Moreover, the right-hand side becomes

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_1(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Since the two sides of the differential equation agree, $\mathbf{Y}_1(t)$ is a solution.

Similarly, if we substitute $\mathbf{Y}_2(t)$ in the left-hand side of the differential equation, we get

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Moreover, the right-hand side is

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_2(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{2t} \\ e^{2t} + e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Since the two sides of the differential equation also agree for this function, $\mathbf{Y}_2(t)$ is another solution.

(b) At $t = 0$, $\mathbf{Y}(0) = (-2, -1)$. By the Linearity Principle, any linear combination of two solutions is also a solution. Hence, we solve the given initial-value problem with a function of the form $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ where k_1 and k_2 are constants determined by the initial value. That is, we determine k_1 and k_2 via

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = \mathbf{Y}(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

We get

$$k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous linear equations

$$\begin{cases} k_2 = -2 \\ k_1 + k_2 = -1. \end{cases}$$

From the first equation, we have $k_2 = -2$. Then from the second equation, we obtain $k_1 = 1$. Therefore, the solution to the initial-value problem is

$$\begin{aligned} \mathbf{Y}(t) &= \mathbf{Y}_1(t) - 2\mathbf{Y}_2(t) \\ &= \begin{pmatrix} 0 \\ e^t \end{pmatrix} - 2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} \\ e^t - 2e^{2t} \end{pmatrix}. \end{aligned}$$

Note that (as always) we can check our calculations directly. By direct evaluation, we know that $\mathbf{Y}(0) = (-2, -1)$. Moreover, we can check that $\mathbf{Y}(t)$ satisfies the differential equation. The left-hand side of the differential equation is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4e^{2t} \\ e^t - 4e^{2t} \end{pmatrix},$$

and the right-hand side of the differential equation is

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{Y}(t) &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2e^{2t} \\ e^t - 4e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -4e^{2t} \\ e^t - 4e^{2t} \end{pmatrix}. \end{aligned}$$

Since the left-hand side and the right-hand side agree, the function $\mathbf{Y}(t)$ is a solution to the differential equation, and since it assumes the given initial value, this function is the desired solution to the initial-value problem. The Uniqueness Theorem says that this function is the only solution to the initial-value problem.

26. **(a)** Substitute $\mathbf{Y}_1(t)$ into the differential equation and compare the left-hand side to the right-hand side. On the left-hand side, we have

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix},$$

and on the right-hand side, we have

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Since the two sides agree, we know that $\mathbf{Y}_1(t)$ is a solution.

For $\mathbf{Y}_2(t)$,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -2e^{-4t} - 2e^{-4t} \\ 2e^{-4t} - 10e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}.$$

Since the two sides agree, the function $\mathbf{Y}_2(t)$ is also a solution.

Both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, and we proceed to the next part of the exercise.

(b) Note that $\mathbf{Y}_1(0) = (1, 1)$ and $\mathbf{Y}_2(0) = (1, 2)$. These vectors are not on the same line through the origin, so the initial conditions are linearly independent. If the initial conditions are linearly independent, then the solutions must also be linearly independent. Since the two solutions are linearly independent, we proceed to part (c) of the exercise.

(c) We must find constants k_1 and k_2 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

In other words, the constants k_1 and k_2 must satisfy the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ k_1 + 2k_2 = 3. \end{cases}$$

It follows that $k_1 = 1$ and $k_2 = 1$. Hence, the required solution is

$$\mathbf{Y}_1(t) + \mathbf{Y}_2(t) = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}.$$

34. **(a)** If $\mathbf{Y}(t) = (t, t^2/2)$, then $x(t) = t$ and $y(t) = t^2/2$. Then $dx/dt = 1$, and $dy/dt = t = x$. So $\mathbf{Y}(t)$ satisfies the differential equation.

(b) For $2\mathbf{Y}(t)$, we have $x(t) = 2t$, and $y(t) = t^2$. In this case, we need only consider $dx/dt = 2$ to see that the function is not a solution to the system.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt}y_2 + x_1\frac{dy_2}{dt} - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1$$

and that

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

1. **(a)** The characteristic polynomial is

$$(3-\lambda)(-2-\lambda)=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=3$.

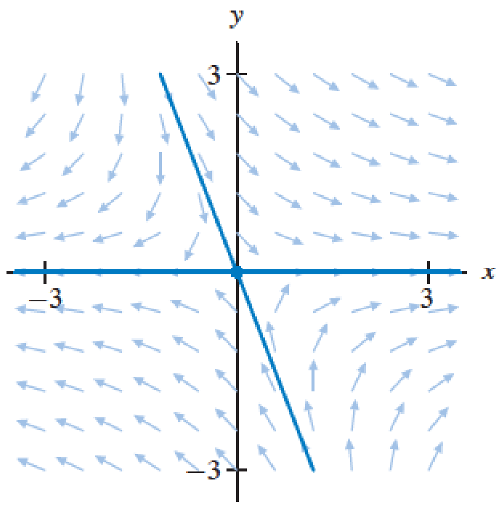
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} 3x_1+2y_1=-2x_1 \\ -2y_1=-2y_1 \end{cases}$$

and obtain $5x_1=-2y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2=3$ must satisfy the equation $y_2=0$.

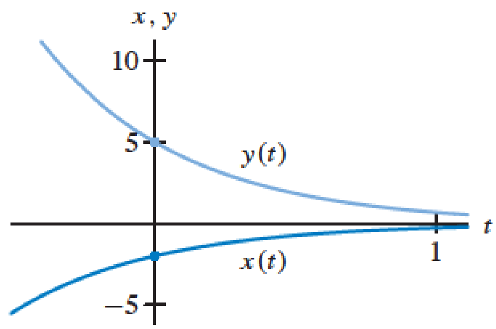
(c)



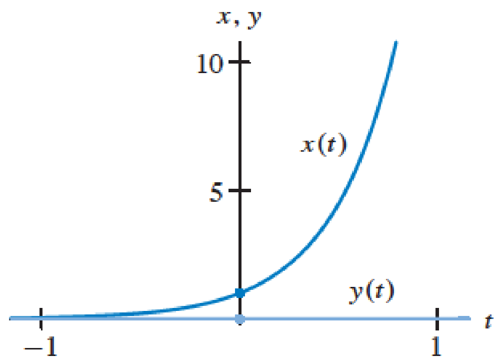
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (-2, 5)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 0)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. **(a)** The characteristic polynomial is

$$(-4-\lambda)(-3-\lambda)-2=\lambda^2+7\lambda+10=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-5$.

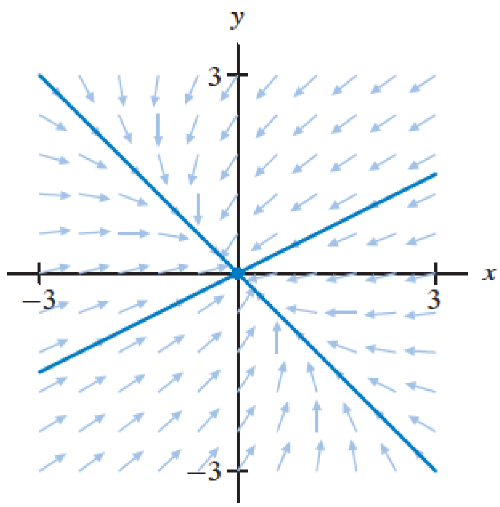
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} -4x_1-2y_1=-2x_1 \\ -x_1-3y_1=-2y_1 \end{cases}$$

and obtain $y_1=-x_1$.

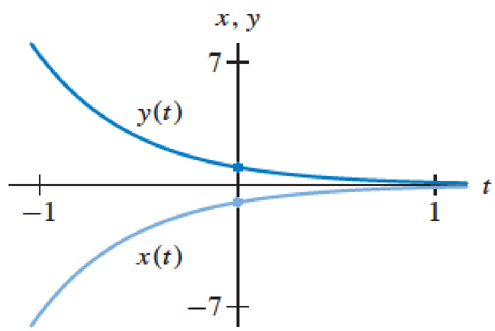
Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=2y_2$ for $\lambda_2=-5$.

(c)

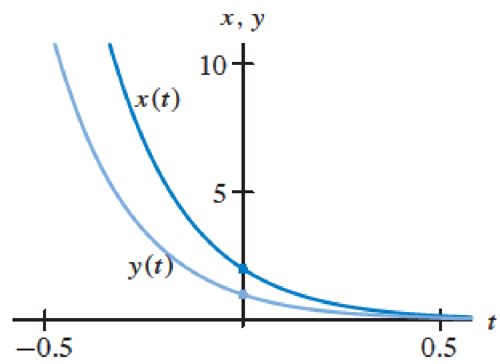


(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (2, 1)$.
Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda) - 36 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$.

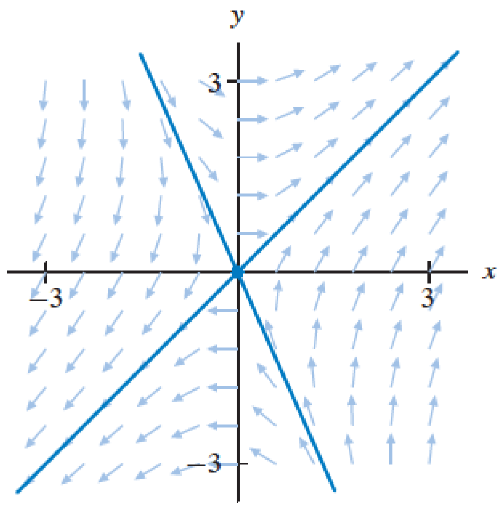
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$, we solve the system of equations

$$\begin{cases} 5x_1 + 4y_1 = -4x_1 \\ 9x_1 = -4y_1 \end{cases}$$

and obtain $9x_1 = -4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ must satisfy the equation $y_2 = x_2$.

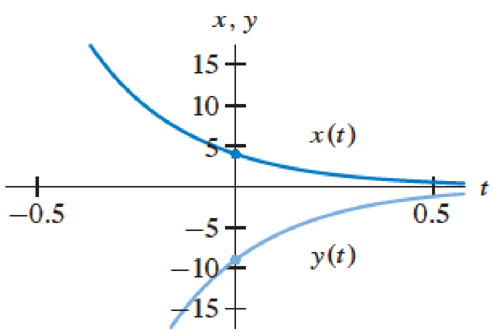
(c)



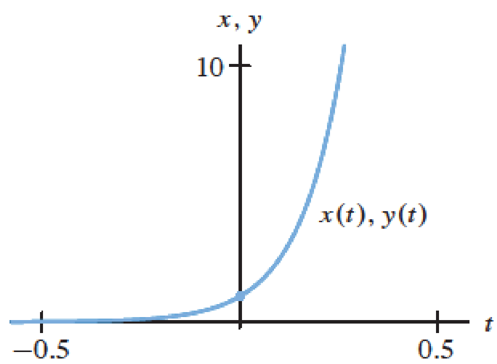
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (4, -9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. **(a)** The characteristic polynomial is

$$(3-\lambda)(-\lambda)-4=\lambda^2-3\lambda-4=(\lambda-4)(\lambda+1)=0,$$

and therefore the eigenvalues are $\lambda_1=-1$ and $\lambda_2=4$.

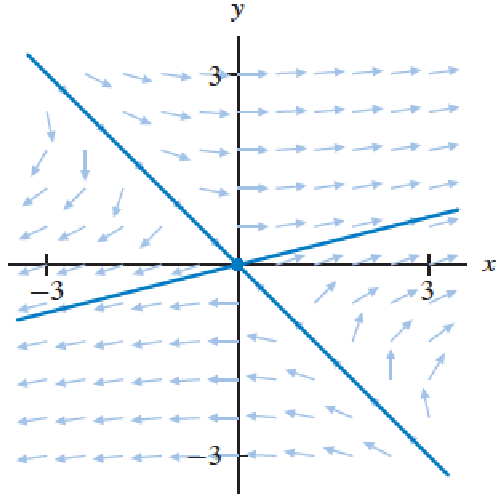
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-1$, we solve the system of equations

$$\begin{cases} 3x_1+4y_1=-x_1 \\ x_1=-y_1 \end{cases}$$

and obtain $y_1=-x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=4y_2$ for $\lambda_2=4$.

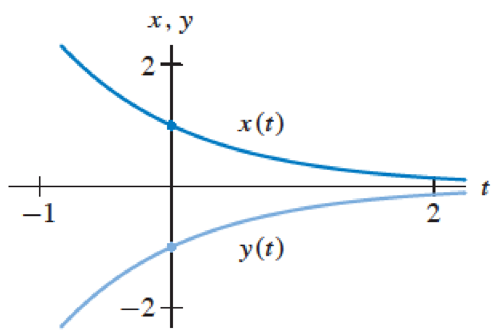
(c)



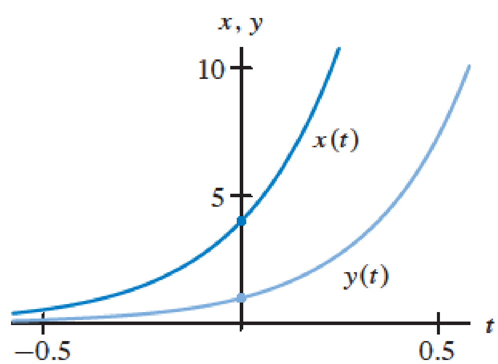
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(1,-1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(4,1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t)=e^{4t}\begin{pmatrix}4\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t)=k_1e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}+k_2e^{4t}\begin{pmatrix}4\\1\end{pmatrix}$$

14. The characteristic polynomial is

$$(4-\lambda)(1-\lambda)+2=\lambda^2-5\lambda+6=0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 4x_1 - 2y_1 = 3x_1 \\ x_1 + y_1 = 3y_1 \end{cases}$$

and obtain

$$x_1 = 2y_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (2, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (1, 1)$ for $\lambda_2 = 2$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 2k_1 + k_2 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = -1$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (2, 1)$. Since this initial condition is an eigenvector associated to the $\lambda = 3$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-1, -2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 2k_1 + k_2 = -1 \\ k_1 + k_2 = -2. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = -3$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

20. **(a)** The parameters $m = 1, k = 4$, and $b = 5$ yield the second-order equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0.$$

Given $v = dy/dt$, the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -4y - 5v.$$

The characteristic polynomial is $\lambda^2 + 5\lambda + 4$, and the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -1$.

To find the eigenvectors $\mathbf{V}_1 = (y_1, v_1)$ associated to the eigenvalue $\lambda_1 = -4$, we solve the system of equations.

$$\begin{cases} v_1 = -4y_1 \\ -4y_1 - 5v_1 = -4v_1 \end{cases}$$

and obtain $v_1 = -4y_1$. Thus, one eigenvector for $\lambda_1 = -4$ is $\mathbf{V}_1 = (1, -4)$.

By the same procedure, we can find the eigenvector $\mathbf{V}_2 = (1, -1)$ for the eigenvalue $\lambda_2 = -1$.

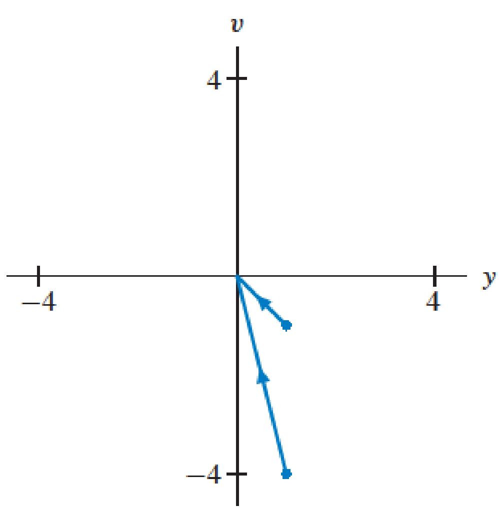
(b) Therefore the solution $\mathbf{Y}_1(t)$ that satisfies $\mathbf{Y}_1(0) = \mathbf{V}_1$ is

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

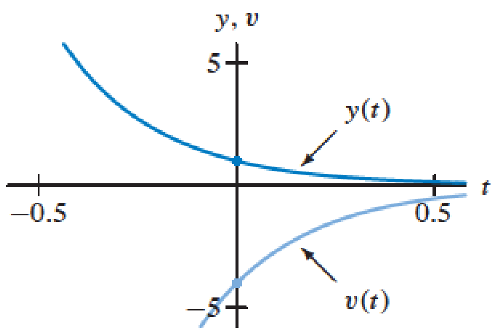
The solution $\mathbf{Y}_2(t)$ that satisfies $\mathbf{Y}_2(0) = \mathbf{V}_2$ is

$$\mathbf{Y}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

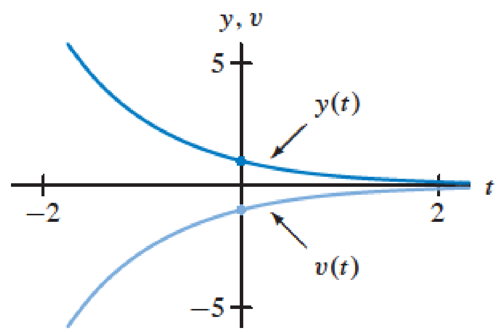
(c)



(d)



The $y(t)$- and $v(t)$-graphs for $\mathbf{Y}_1(t)$.



The $y(t)$- and $v(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The first initial condition $(y_0, v_0) = (1, -4)$ represents a solution whose initial position is 1 unit away from the equilibrium position and whose initial velocity is -4. Note that the solution tends toward the equilibrium point at the origin. Moreover, $y(t)$ is decreasing toward 0, and $v(t)$ is increasing toward 0. Therefore, the mass moves toward the equilibrium position monotonically, and its speed decreases as it approaches the equilibrium position. The mass does not oscillate about the equilibrium position.

The second initial condition $(y_0, v_0) = (1, -1)$ represents a solution whose initial position is 1 unit away from the equilibrium position and whose initial velocity is -1. The behavior of this solution is similar to the first solution.