3. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + (3\alpha \cos 2t + 3\beta \sin 2t) \\ &= (3\alpha + 2\beta)\cos 2t + (3\beta - 2\alpha)\sin 2t\end{aligned}$$

Consequently, we must have

$$(3\alpha + 2\beta)\cos 2t + (3\beta - 2\alpha)\sin 2t = 4\cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 3\alpha + 2\beta = 4 \\ 3\beta - 2\alpha = 0. \end{cases}$$

Hence, $\alpha = 12/13$ and $\beta = 8/13$. The general solution is

$$y(t) = ke^{-3t} + \tfrac{12}{13}\cos 2t + \tfrac{8}{13}\sin 2t.$$

6. The general solution of the associated homogeneous equation is $y_h(t) = ke^{t/2}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha t e^{t/2}$ rather than $\alpha e^{t/2}$ because $\alpha e^{t/2}$ is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - \frac{y_p}{2} &= \alpha e^{t/2} + \frac{\alpha}{2} t e^{t/2} - \frac{\alpha t e^{t/2}}{2} \\ &= \alpha e^{t/2}.\end{aligned}$$

Consequently, we must have $\alpha = 4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{t/2} + 4te^{t/2}.$$

12. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha t e^{2t}$ rather than αe^{2t} because αe^{2t} is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= \alpha e^{2t} + 2\alpha t e^{2t} - 2\alpha t e^{2t} \\ &= \alpha e^{2t}.\end{aligned}$$

Consequently, we must have $\alpha = 7$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} + 7te^{2t}.$$

Note that $y(0) = k$, so the solution to the initial-value problem is

$$y(t) = 3e^{2t} + 7te^{2t} = (7t + 3)e^{2t}.$$

17. **(a)** We compute

$$\frac{dy_1}{dt} = \frac{1}{(1-t)^2} = (y_1(t))^2$$

to see that $y_1(t)$ is a solution.

(b) We compute

$$\frac{dy_2}{dt} = 2\frac{1}{(1-t)^2} \neq (y_2(t))^2$$

to see that $y_2(t)$ is not a solution.

(c) The equation $dy/dt = y^2$ is not linear. It contains y^2.

22. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side t^3 and one for the right-hand side $\sin 3t$.

With the right-hand side t^3, we are tempted to guess that there is a solution of the form at^3, but there isn't. Instead we guess a solution of the form

$$y_{p_1}(t) = at^3 + bt^2 + ct + d.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + y_{p_1} &= 3at^2 + 2bt + c + at^3 + bt^2 + ct + d \\ &= at^3 + (3a + b)t^2 + (2b + c)t + (c + d)\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} a = 1 \\ 3a + b = 0 \\ 2b + c = 0 \\ c + d = 0. \end{cases}$$

We get $a = 1, b = -3, c = 6$, and $d = -6$.

With the right-hand side $\sin 3t$, we guess a solution of the form

$$y_{p_2}(t) = \alpha \cos 3t + \beta \sin 3t.$$

Then

$$\begin{aligned}\frac{dy_{p_2}}{dt} + y_{p_1} &= -3\alpha \sin 3t + 3\beta \cos 3t + \alpha \cos 3t + \beta \sin 3t \\ &= (\alpha + 3\beta) \cos 3t + (-3\alpha + \beta) \sin 3t.\end{aligned}$$

Then y_{p_2} is a solution if

$$\begin{cases} \alpha + 3\beta = 0 \\ -3\alpha + \beta = 1. \end{cases}$$

We get $\alpha = -3/10$ and $\beta = 1/10$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$, so the general solution of the original equation is

$$ke^{-t} + t^3 - 3t^2 + 6t - 6 - \tfrac{3}{10} \cos 3t + \tfrac{1}{10} \sin 3t.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$k - 6 - \tfrac{3}{10} = 0.$$

Hence, $k = 63/10$.

31. Step 1: *Before retirement*

First we calculate how much money will be in her retirement fund after 30 years. The differential equation modeling the situation is

$$\frac{dy}{dt} = .07y + 5{,}000,$$

where $y(t)$ represents the fund's balance at time t.

The general solution of the homogeneous equation is $y_h(t) = ke^{0.07t}$.

To find a particular solution, we observe that the nonhomogeneous equation is autonomous and that it has an equilibrium solution at $y = -5{,}000/0.07 \approx -71{,}428.57$. We can use this equilibrium solution as the particular solution. (It is the solution we would have computed if we had guessed a constant solution). We obtain

$$y(t) = ke^{0.07t} - 71{,}428.57.$$

From the initial condition, we see that $k = 71{,}428.57$, and

$$y(t) = 71{,}428.57(e^{0.07t} - 1).$$

Letting $t = 30$, we compute that the fund contains $\approx \$511{,}869.27$ after 30 years.

Step 2: *After retirement*

We need a new model for the remaining years since the professor is withdrawing rather than depositing. Since she withdraws at a rate of \$3,000 per month (\$36,000 per year), we write

$$\frac{dy}{dt} = .07y - 36{,}000,$$

where we continue to measure time t in years.

Again, the solution of the homogeneous equation is $y_h(t) = ke^{0.07t}$.

To find a particular solution of the nonhomogeneous equation, we note that the equation is autonomous and that it has an equilibrium at $y = 36{,}000/0.07 \approx 514{,}285.71$. Hence, we may take the particular solution to be this equilibrium solution. (Again, this solution is what we would have computed if we had guessed a constant function for y_p.)

The general solution is

$$y(t) = ke^{0.07t} + 514{,}285.71.$$

In this case, we have the initial condition $y(0) = 511{,}869.27$ since now $y(t)$ is the amount in the fund t years after she retires. Solving $511{,}869.27 = k + 514{,}285.71$, we get $k = -2{,}416.44$. The solution in this case is

$$y(t) = -2{,}416.44e^{0.07t} + 514{,}285.71.$$

Finally, we wish to know when her money runs out. That is, at what time t is $y(t) = 0$? Solving

$$y(t) = -2{,}416.44e^{0.07t} + 514{,}285.71 = 0$$

yields $t \approx 76.58$ years (approximately 919 months).

3. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{1+t} = t^2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/(1+t))\,dt} = e^{\ln(1+t)} = 1 + t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(1+t)\frac{dy}{dt} + y = (1+t)t^2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d((1+t)y)}{dt} = t^3 + t^2,$$

and integrating both sides with respect to t, we obtain

$$(1+t)y = \frac{t^4}{4} + \frac{t^3}{3} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{3t^4 + 4t^3 + 12c}{12(t+1)}.$$

4. We rewrite the equation in the form

$$\frac{dy}{dt} + 2ty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{t^2}y)}{dt} = 4,$$

and integrating both sides with respect to t, we obtain

$$e^{t^2}y = 4t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

8. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{1}{t+1}\, y = 4t^2 + 4t$$

and note that the integrating factor is

$$\mu(t) = e^{\int (-1/(t+1))\, dt} = e^{-\ln(t+1)} = \left(e^{\ln((t+1)^{-1})}\right) = \frac{1}{t+1}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t+1}\frac{dy}{dt} - \frac{1}{(t+1)^2}\, y = \frac{4t^2 + 4t}{t+1}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t+1}\right) = 4t.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{t+1} = 2t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (2t^2 + c)(t+1) = 2t^3 + 2t^2 + ct + c.$$

To find the solution that satisfies the initial condition $y(1) = 10$, we evaluate the general solution at $t = 1$ and obtain $c = 3$. The desired solution is

$$y(t) = 2t^3 + 2t^2 + 3t + 3.$$

11. Note that the integrating factor is

$$\mu(t) = e^{\int -(2/t)\,dt} = e^{-2\int (1/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3} = 2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 2,$$

and integrating both sides with respect to t, we obtain

$$\frac{y}{t^2} = 2t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 2t^3 + ct^2.$$

To find the solution that satisfies the initial condition $y(-2) = 4$, we evaluate the general solution at $t = -2$ and obtain

$$-16 + 4c = 4.$$

Hence, $c = 5$, and the desired solution is

$$y(t) = 2t^3 + 5t^2.$$

12. Note that the integrating factor is

$$\mu(t) = e^{\int (-3/t)\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-3}\frac{dy}{dt} - 3t^{-4}y = 2e^{2t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-3}y)}{dt} = 2e^{2t},$$

and integrating both sides with respect to t, we obtain

$$t^{-3}y = e^{2t} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^3(e^{2t} + c).$$

To find the solution that satisfies the initial condition $y(1) = 0$, we evaluate the general solution at $t = 1$ and obtain $c = -e^2$. The desired solution is

$$y(t) = t^3(e^{2t} - e^2).$$

21. (**a**) The integrating factor is

$$\mu(t) = e^{0.4t}.$$

Multiplying both sides of the differential equation by $\mu(t)$ and collecting terms, we obtain

$$\frac{d(e^{0.4t}v)}{dt} = 3e^{0.4t}\cos 2t.$$

Integrating both sides with respect to t yields

$$e^{0.4t}v = \int 3e^{0.4t}\cos 2t\,dt.$$

To calculate the integral on the right-hand side, we must integrate by parts twice.

For the first integration, we pick $u_1(t) = \cos 2t$ and $v_1(t) = e^{0.4t}$. Using the fact that $0.4 = 2/5$, we get

$$\int e^{0.4t}\cos 2t\,dt = \tfrac{5}{2}\,e^{0.4t}\cos 2t + 5\int e^{0.4t}\sin 2t\,dt.$$

For the second integration, we pick $u_2(t) = \sin 2t$ and $v_2(t) = e^{0.4t}$. We get

$$\int e^{0.4t}\sin 2t\,dt = \tfrac{5}{2}e^{0.4t}\sin 2t - 5\int e^{0.4t}\cos 2t\,dt.$$

Combining these results yields

$$\int e^{0.4t}\cos 2t\,dt = \tfrac{5}{2}\,e^{0.4t}\cos 2t + \tfrac{25}{2}\,e^{0.4t}\sin 2t - 25\int e^{0.4t}\cos 2t\,dt.$$

Solving for $\int e^{0.4t}\cos 2t\,dt$, we have

$$\int e^{0.4t}\cos 2t\,dt = \frac{5\,e^{0.4t}\cos 2t + 25\,e^{0.4t}\sin 2t}{52}.$$

To obtain the general solution, we multiply this integral by 3, add the constant of integration, and solve for v. We obtain the general solution

$$v(t) = ke^{-0.4t} + \tfrac{15}{52}\cos 2t + \tfrac{75}{52}\sin 2t.$$

(b) The solution of the associated homogeneous equation is

$$v_h(t) = e^{-0.4t}.$$

We guess

$$v_p(t) = \alpha \cos 2t + \beta \sin 2t$$

for the a solution to the nonhomogeneous equation and solve for α and β. Substituting this guess into the differential equation, we obtain

$$-2\alpha \sin 2t + 2\beta \cos 2t + 0.4\alpha \cos 2t + 0.4\beta \sin 2t = 3 \cos 2t.$$

Collecting sine and cosine terms, we get the system of equations

$$\begin{cases} -2\alpha + 0.4\beta = 0 \\ 0.4\alpha + 2\beta = 3. \end{cases}$$

Using the fact that $0.4 = 2/5$, we solve this system of equations and obtain

$$\alpha = \tfrac{15}{52} \quad \text{and} \quad \beta = \tfrac{75}{52}.$$

The general solution of the original nonhomogeneous equation is

$$v(t) = ke^{-0.4t} + \tfrac{15}{52} \cos 2t + \tfrac{75}{52} \sin 2t.$$

Both methods require quite a bit of computation. If we use an integrating factor, we must do a complicated integral, and if we use the guessing technique, we have to be careful with our algebra.