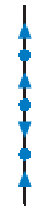
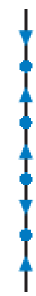
31. The function $f(y)$ has three zeros. We denote them as y_1, y_2, and y_3, where $y_1 < 0 < y_2 < y_3$. So the differential equation $dy/dt = f(y)$ has three equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, $f(y) < 0$ if $y_1 < y < y_2$, and $f(y) > 0$ if $y_2 < y < y_3$ or if $y > y_3$. Hence y_1 is a sink, y_2 is a source, and y_3 is a node.

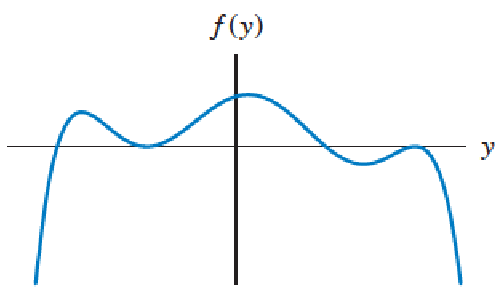


32. The function $f(y)$ has four zeros, which we denote $y_1, \ldots, y_4$ where $y_1 < 0 < y_2 < y_3 < y_4$. So the differential equation $dy/dt = f(y)$ has four equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, if $y_2 < y < y_3$, or if $y_3 < y < y_4$; and $f(y) < 0$ if $y_1 < y < y_2$ or if $y > y_4$. Hence y_1 is a sink, y_2 is a source, y_3 is a node, and y_4 is a sink.



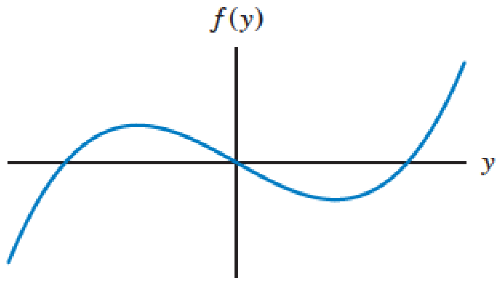
34. Since there are four equilibrium points, the graph of $f(y)$ must touch the y-axis at four distinct numbers y_1, y_2, y_3, and y_4. We assume that $y_1 < y_2 < y_3 < y_4$. Since the arrows point up only if $y_1 < y < y_2$ or if $y_2 < y < y_3$, we must have $f(y) > 0$ for $y_1 < y < y_2$ and for $y_2 < y < y_3$. Moreover, $f(y) < 0$ if $y < y_1$, if $y_3 < y < y_4$, or if $y > y_4$. Therefore, the graph of f crosses the y-axis at y_1 and y_3, but it is tangent to the y-axis at y_2 and y_4.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



35. Since there are three equilibrium points (one appearing to be at $y = 0$), the graph of $f(y)$ must touch the y-axis at three numbers y_1, y_2, and y_3. We assume that $y_1 < y_2 = 0 < y_3$. Since the arrows point down for $y < y_1$ and $y_2 < y < y_3$, $f(y) < 0$ for $y < y_1$ and for $y_2 < y < y_3$. Similarly, $f(y) > 0$ if $y_1 < y < y_2$ and if $y > y_3$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



37. **(a)** This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi), and (viii) have exactly these equilibria. There exists a node at $y = 0$. Only equations (iv) and (viii) have a node at $y = 0$. Moreover, for this phase line, $dy/dt < 0$ for $y > 1$. Only equation (viii) satisfies this property. Consequently, the phase line corresponds to equation (viii).

(b) This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi) and (viii) have exactly these equilibria. Moreover, for this phase line, $dy/dt > 0$ for $y > 1$. Only equations (iv) and (vi) satisfy this property. Lastly, $dy/dt > 0$ for $y < 0$. Only equation (vi) satisfies this property. Consequently, the phase line corresponds to equation (vi).

(c) This phase line has an equilibrium point at $y = 3$. Only equations (i) and (v) have this equilibrium point. Moreover, this phase line has another equilibrium point at $y = 0$. Only equation (i) satisfies this property. Consequently, the phase line corresponds to equation (i).

(d) This phase line has an equilibrium point at $y = 2$. Only equations (iii) and (vii) have this equilibrium point. Moreover, there exists a node at $y = 0$. Only equation (vii) satisfies this property. Consequently, the phase line corresponds to equation (vii).

38. **(a)** Because $f(y)$ is continuous we can use the Intermediate Value Theorem to say that there must be a zero of $f(y)$ between -10 and 10. This value of y is an equilibrium point of the differential equation. In fact, $f(y)$ must cross from positive to negative, so if there is a single equilibrium point, it must be a sink (see part (b)).

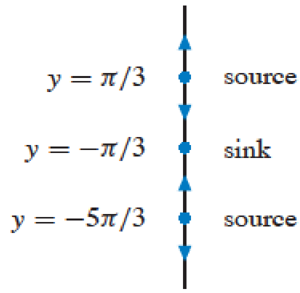
(b) We know that $f(y)$ must cross the y-axis between -10 and 10. Moreover, it must cross from positive to negative because $f(-10)$ is positive and $f(10)$ is negative. Where $f(y)$ crosses the y-axis from positive to negative, we have a sink. If $y = 1$ is a source, then crosses the y-axis from negative to positive at $y = 1$. Hence, $f(y)$ must cross the y-axis from positive to negative at least once between $y = -10$ and $y = 1$ and at least once between $y = 1$ and $y = 10$. There must be at least one sink in each of these intervals. (We need the assumption that the number of equilibrium points is finite to prevent cases where $f(y) = 0$ along an entire interval.)

40. **(a)** The equilibrium points of $d\theta/dt = f(\theta)$ are the numbers θ where $f(\theta) = 0$. For

$$f(\theta) = 1 - \cos\theta + (1 + \cos\theta)\left(-\tfrac{1}{3}\right) = \tfrac{2}{3}(1 - 2\cos\theta),$$

the equilibrium points are $\theta = 2\pi n \pm \pi/3$, where $n = 0, \pm 1, \pm 2, \ldots$.

(b) The sign of $d\theta/dt$ alternates between positive and negative at successive equilibrium points. It is negative for $-\pi/3 < \theta < \pi/3$ and positive for $\pi/3 < \theta < 5\pi/3$. Therefore, $\pi/3 = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



43. **(a)** Because the first and second derivative are zero at y_0 and the third derivative is positive, Taylor's Theorem implies that the function $f(y)$ is approximately equal to

$$\frac{f'''(y_0)}{3!}(y - y_0)^3$$

for y near y_0. Since $f'''(y_0) > 0, f(y)$ is increasing near y_0. Hence, y_0 is a source.

(b) Just as in part (a), we see that $f(y)$ is decreasing near y_0, so y_0 is a sink.

(c) In this case, we can approximate $f(y)$ near y_0 by

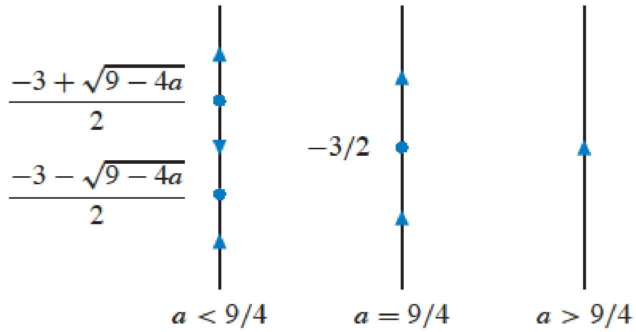
$$\frac{f''(y_0)}{2!}(y - y_0)^2.$$

Since the second derivative of $f(y)$ at y_0 is assumed to be positive, $f(y)$ is positive on both sides of y_0 for y near y_0. Hence y_0 is a node.

2. The equilibrium points occur at solutions of $dy/dt = y^2 + 3y + a = 0$. From the quadratic formula, we have

$$y = \frac{-3 \pm \sqrt{9 - 4a}}{2}.$$

Hence, the bifurcation value of a is $9/4$. For $a < 9/4$, there are two equilibria, one source and one sink. For $a = 9/4$, there is one equilibrium which is a node, and for $a > 9/4$, there are no equilibria.



Phase lines for $a < 9/4$, $a = 9/4$, and $a > 9/4$.

3. The equilibrium points occur at solutions of $dy/dt = y^2 - ay + 1 = 0$. From the quadratic formula, we have

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

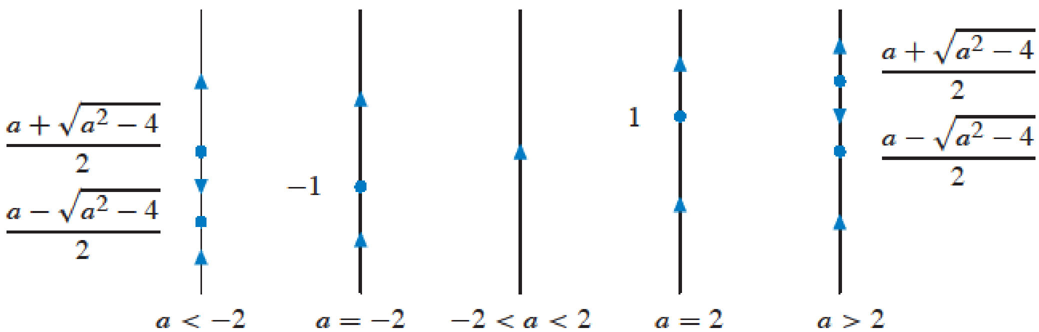
If $-2 < a < 2$, then $a^2 - 4 < 0$, and there are no equilibrium points. If $a > 2$ or $a < -2$, there are two equilibrium points. For $a = \pm 2$, there is one equilibrium point at $y = a/2$. The bifurcations occur at $a = \pm 2$.

To draw the phase lines, note that:

- For $-2 < a < 2$, $dy/dt = y^2 - ay + 1 > 0$, so the solutions are always increasing.
- For $a = 2$, $dy/dt = (y - 1)^2 \geq 0$, and $y = 1$ is a node.
- For $a = -2$, $dy/dt = (y + 1)^2 \geq 0$, and $y = -1$ is a node.
- For $a < -2$ or $a > 2$, let

$$y_1 = \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad y_2 = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $dy/dt < 0$ if $y_1 < y < y_2$, and $dy/dt > 0$ if $y < y_1$ or $y > y_2$.



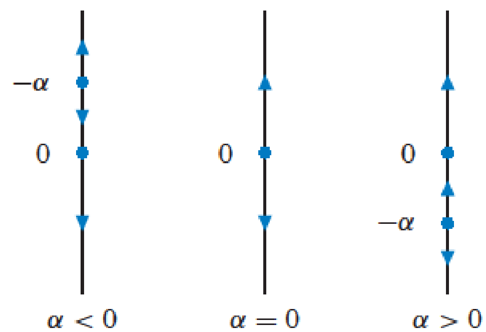
The five possible phase lines.

4. The equilibrium points occur at solutions of $dy/dt = y^3 + \alpha y^2 = 0$. For $\alpha = 0$, there is one equilibrium point, $y = 0$. For $\alpha \neq 0$, there are two equilibrium points, $y = 0$ and $y = -\alpha$. Thus, $\alpha = 0$ is a bifurcation value.

To draw the phase lines, note that:

- If $\alpha < 0$, $dy/dt > 0$ only if $y > -\alpha$.
- If $\alpha = 0$, $dy/dt > 0$ if $y > 0$, and $dy/dt < 0$ if $y < 0$.
- If $\alpha > 0$, $dy/dt < 0$ only if $y < -\alpha$.

Hence, as α increases from negative to positive, the source at $y = -\alpha$ moves from positive to negative as it "passes through" the node at $y = 0$.



11. For $\alpha = 0$, there are three equilibria. There is a sink to the left of $y = 0$, a source at $y = 0$, and a sink to the right of $y = 0$.

As α decreases, the source and sink on the right move together. A bifurcation occurs at $\alpha \approx -2$. At this bifurcation value, there is a sink to the left of $y = 0$ and a node to the right of $y = 0$. For α below this bifurcation value, there is only the sink to the left of $y = 0$.

As α increases from zero, the sink to the left of $y = 0$ and the source move together. There is a bifurcation at $\alpha \approx 2$ with a node to the left of $y = 0$ and a sink to the right of $y = 0$. For α above this bifurcation value, there is only the sink to the right of $y = 0$.

13. **(a)** Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). Since $y = 0$ is the only equilibrium point for $A < 0$, this only corresponds to equation (iii).

(b) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt < 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (v).

(c) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt > 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (iv).

(d) Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). The phase lines corresponding to $A > 0$ only have two nonnegative equilibrium points. Consequently, the bifurcation diagram corresponds to equation (i).

18. **(a)** For all $C \geq 0$, the equation has a source at $P = C/k$, and this is the only equilibrium point. Hence all of the phase lines are qualitatively the same, and there are no bifurcation values for C.

(b) If $P(0) > C/k$, the corresponding solution $P(t) \to \infty$ at an exponential rate as $t \to \infty$, and if $P(0) < C/k$, $P(t) \to -\infty$, passing through "extinction" ($P = 0$) after a finite time.

19. **(a)** A model of the fish population that includes fishing is

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L,$$

where L is the number of licenses issued. The coefficient of 3 represents the average catch of 3 fish per year. As L is increased, the two equilibrium points for $L = 0$ (at $P = 0$ and $P = 100$) will move together. If L is sufficiently large, there are no equilibrium points. Hence we wish to pick L as large as possible so that there is still an equilibrium point present. In other words, we want the bifurcation value of L. The bifurcation value of L occurs if the equation

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L = 0$$

has just one solution for P in terms of L. Using the quadratic formula, we see that there is exactly one equilibrium point if $L = 50/3$. Since this value of L is not an integer, the largest number of licenses that should be allowed is 16.

(b) If we allow the fish population to come to equilibrium then the population will be at the carrying capacity, which is $P = 100$ if $L = 0$. If we then allow 16 licenses to be issued, we expect that the population is a solution to the new model with $L = 16$ and initial population $P = 100$. The model becomes

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 48,$$

which has a source at $P = 40$ and a sink at $P = 60$.

Thus, any initial population greater than 40 when fishing begins tends to the equilibrium level $P = 60$. If the initial population of fish was less than 40 when fishing begins, then the model predicts that the population will decrease to zero in a finite amount of time.

(c) The maximum "number" of licenses is $16\frac{2}{3}$. With $L = 16\frac{2}{3}$, there is an equilibrium at $P = 50$. This equilibrium is a node, and if $P(0) > 50$, the population will approach 50 as t increases. However, it is dangerous to allow this many licenses since an unforeseen event might cause the death of a few extra fish. That event would push the number of fish below the equilibrium value of $P = 50$. In this case, $dP/dt < 0$, and the population decreases to extinction.

If, however, we restrict to $L = 16$ licenses, then there are two equilibria, a sink at $P = 60$ and source at $P = 40$. As long as $P(0) > 40$, the population will tend to 60 as t increases. In this case, we have a small margin of safety. If $P \approx 60$, then it would have to drop to less than 40 before the fish are in danger of extinction.

4. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned} \frac{dy_p}{dt} - 2y_p &= -2\alpha \sin 2t + 2\beta \cos 2t - 2(\alpha \cos 2t + \beta \sin 2t) \\ &= (2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t. \end{aligned}$$

Consequently, we must have

$$(2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t = \sin 2t$$

for $y_p(t)$ to be a solution, that is, we must solve

$$\begin{cases} -2\alpha - 2\beta = 1 \\ -2\alpha + 2\beta = 0. \end{cases}$$

Hence, $\alpha = -1/4$ and $\beta = -1/4$. The general solution of the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{1}{4} \cos 2t - \tfrac{1}{4} \sin 2t.$$

5. The general solution to the associated homogeneous equation is $y_h(t) = ke^{3t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{3t}$ rather than αe^{3t} because αe^{3t} is a solution of the homogeneous equation. Then

$$\begin{aligned} \frac{dy_p}{dt} - 3y_p &= \alpha e^{3t} + 3\alpha te^{3t} - 3\alpha te^{3t} \\ &= \alpha e^{3t}. \end{aligned}$$

Consequently, we must have $\alpha = -4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{3t} - 4te^{3t}.$$

8. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-2t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha e^{-2t} - 2\alpha e^{-2t} \\ &= -4\alpha e^{-2t}.\end{aligned}$$

Consequently, we must have $-4\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = -3/4$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{3}{4}e^{-2t}.$$

Since $y(0) = 10$, we have

$$10 = k - \tfrac{3}{4},$$

so $k = 43/4$. The function

$$y(t) = \tfrac{43}{4}e^{2t} - \tfrac{3}{4}e^{-2t}$$

is the solution of the initial-value problem.

10. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + 3\alpha \cos 2t + 3\beta \sin 2t \\ &= (3\alpha + 2\beta)\cos 2t + (-2\alpha + 3\beta)\sin 2t.\end{aligned}$$

Consequently, we must have

$$(3\alpha + 2\beta)\cos 2t + (-2\alpha + 3\beta)\sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 3\alpha + 2\beta = 1 \\ -2\alpha + 3\beta = 0. \end{cases}$$

Hence, $\alpha = 3/13$ and $\beta = 2/13$. The general solution to the differential equation is

$$y(t) = ke^{-3t} + \tfrac{3}{13}\cos 2t + \tfrac{2}{13}\sin 2t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{3}{13}.$$

Since the initial condition is $y(0) = -1$, we see that $k = -16/13$. The desired solution is

$$y(t) = -\tfrac{16}{13}e^{-3t} + \tfrac{3}{13}\cos 2t + \tfrac{2}{13}\sin 2t.$$

20. If $y_p(t) = at^2 + bt + c$, then

$$\frac{dy_p}{dt} + 2y_p = 2at + b + 2at^2 + 2bt + 2c$$
$$= 2at^2 + (2a + 2b)t + (b + 2c).$$

Then $y_p(t)$ is a solution if this quadratic is equal to $3t^2 + 2t - 1$. In other words, $y_p(t)$ is a solution if

$$\begin{cases} 2a = 3 \\ 2a + 2b = 2 \\ b + 2c = -1. \end{cases}$$

From the first equation, we have $a = 3/2$. Then from the second equation, we have $b = -1/2$. Finally, from the third equation, we have $c = -1/4$. The function

$$y_p(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{4}$$

is a solution of the differential equation.

21. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side $t^2 + 2t + 1$ and one for the right-hand side e^{4t}.

With the right-hand side $t^2 + 2t + 1$, we guess a solution of the form

$$y_{p_1}(t) = at^2 + bt + c.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + 2y_{p_1} &= 2at + b + 2(at^2 + bt + c) \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} 2a = 1 \\ 2a + 2b = 2 \\ b + 2c = 1. \end{cases}$$

We get $a = 1/2, b = 1/2$, and $c = 1/4$.

With the right-hand side e^{4t}, we guess a solution of the form

$$y_{p_2}(t) = \alpha e^{4t}.$$

Then

$$\frac{dy_{p_2}}{dt} + 2y_{p_2} = 4\alpha e^{4t} + 2\alpha e^{4t} = 6\alpha e^{4t},$$

and y_{p_2} is a solution if $\alpha = 1/6$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{-2t}$, so the general solution of the original equation is

$$ke^{-2t} + \tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{4} + \tfrac{1}{6}e^{4t}.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$k + \tfrac{1}{4} + \tfrac{1}{6} = 0.$$

Hence, $k = -5/12$.

30. Let $M(t)$ be the amount of money left at time t. Then, we have the initial condition $M(0) = \$70{,}000$. Money is being added to the account at a rate of 1.5% and removed from the account at a rate of \$30,000 per year, so

$$\frac{dM}{dt} = 0.015M - 30{,}000.$$

To find the general solution, we first compute the general solution of the associated homogeneous equation. It is $M_h(t) = ke^{0.015t}$.

To find a particular solution of the nonhomogeneous equation, we observe that the equation is autonomous, and we calculate its equilibrium solution. It is $M(t) = 30{,}000/.015 = \$2{,}000{,}000$ for all t. (This equilibrium solution is what we would have calculated if we had guessed a constant.)

Therefore we have

$$M(t) = 2{,}000{,}000 + ke^{0.015t}.$$

Using the initial condition $M(0) = 70{,}000$, we have

$$2{,}000{,}000 + k = 70{,}000,$$

so $k = -1{,}930{,}000$ and

$$M(t) = 2{,}000{,}000 - 1{,}930{,}000e^{0.015t}.$$

Solving for the value of t when $M(t) = 0$, we have

$$2{,}000{,}000 - 1{,}930{,}000e^{0.015t} = 0,$$

which is equivalent to

$$e^{0.015t} = \frac{2{,}000{,}000}{1{,}930{,}000}.$$

In other words,

$$0.015t = \ln(1.03627),$$

which yields $t \approx 2.375$ years.