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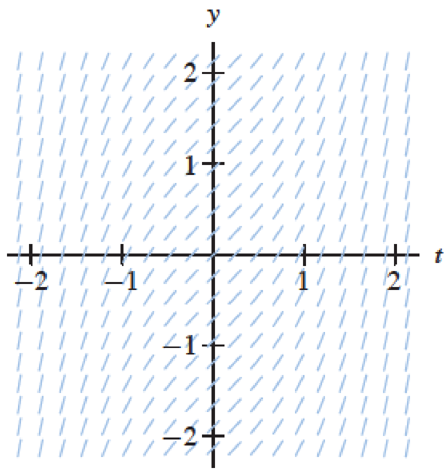

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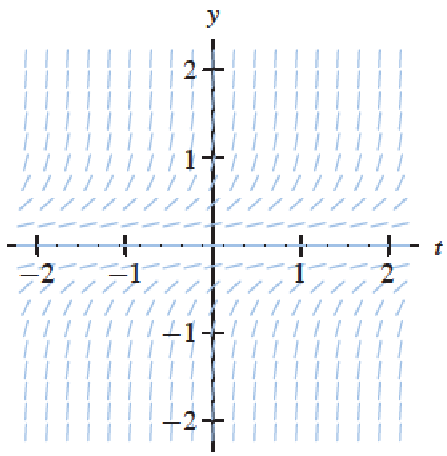
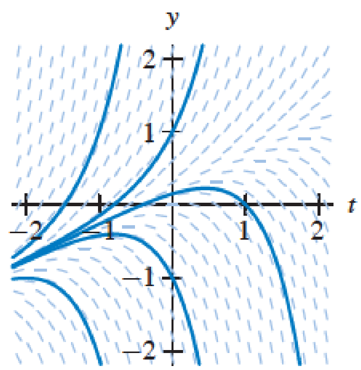
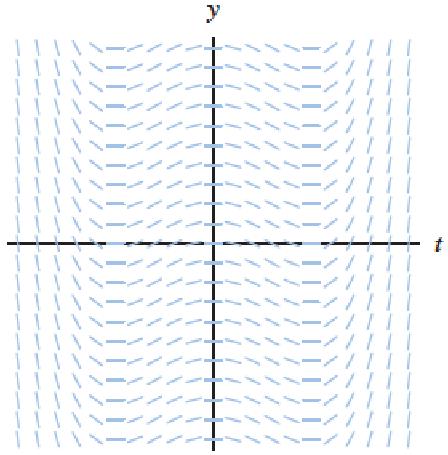


8. **(a)**

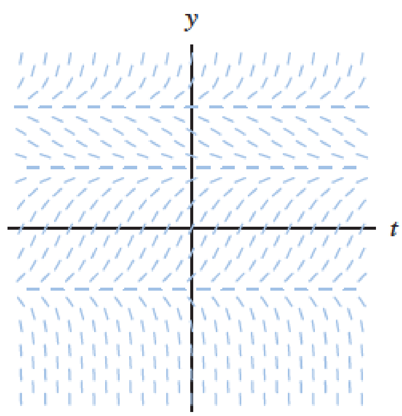


(b) The solution $y(t)$ with $y(0) = 1/2$ increases with $y(t) \to \infty$ as t increases. As t decreases, $y(t) \to -\infty$.

13. The slope field in the ty-plane is constant along vertical lines.



14. Because f depends only on y (the equation is autonomous), the slope field is constant along horizontal lines in the ty-plane. The roots of f correspond to equilibrium solutions. If $f(y) > 0$, the corresponding lines in the slope field have positive slope. If $f(y) < 0$, the corresponding lines in the slope field have negative slope.



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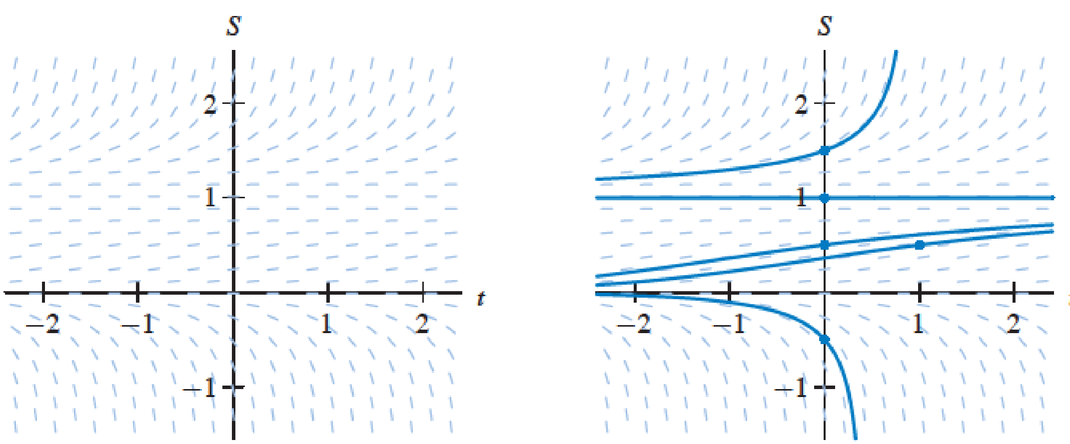


16. **(a)** This slope field is constant along horizontal lines, so it corresponds to an autonomous equation. The autonomous equations are (i), (ii), and (iii). This field does not correspond to equation (ii) because it has the equilibrium solution $y = -1$. The slopes are negative for $y < -1$. Consequently, this field corresponds to equation (iii).

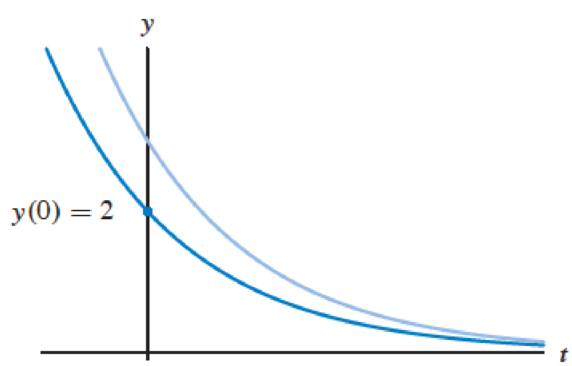
(b) Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation depends only on t. The only choices are equations (iv) and (viii). Since the slopes are negative for $-\sqrt{2} < t < \sqrt{2}$, this slope field corresponds to equation (viii).

(c) This slope field depends both on y and on t, so it can only correspond to equations (v), (vi), or (vii). Since this field has the equilibrium solution $y = 0$, this slope field corresponds to equation (v).

(d) This slope field also depends on both y and on t, so it can only correspond to equations (v), (vi), or (vii). This field does not correspond to equation (v) because $y = 0$ is not an equilibrium solution. Since the slopes are nonnegative for $y > -1$, this slope field corresponds to equation (vi).

18. **(a)** Because the equation is autonomous, the slope field is constant on horizontal lines, so this solution provides enough information to sketch the slope field on the entire upper half plane. Also, if we assume that f is continuous, then the slope field on the line $y = 0$ must be horizontal.

(b) The solution with initial condition $y(0) = 2$ is a translate to the left of the given solution.

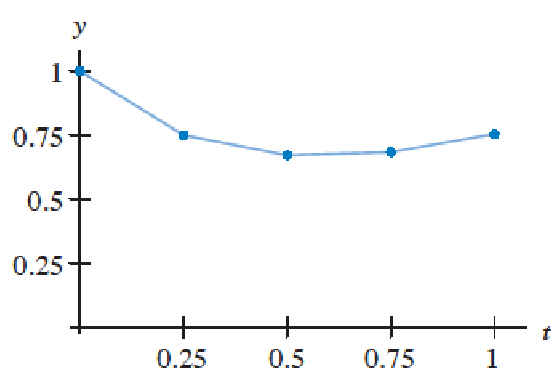


2.

Table 1.2

Results of Euler's method (y_k rounded to two decimal places)

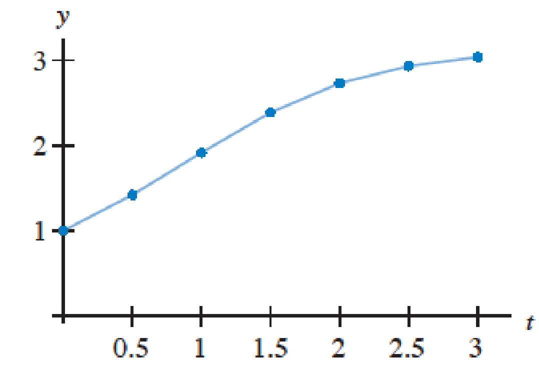
k	t_k	y_k	m_k
0	0	1	-1
1	0.25	0.75	-0.3125
2	0.5	0.67	0.0485
3	0.75	0.68	0.282
4	1.0	0.75	



4.

Table 1.4
Results of Euler's method (to two decimal places)

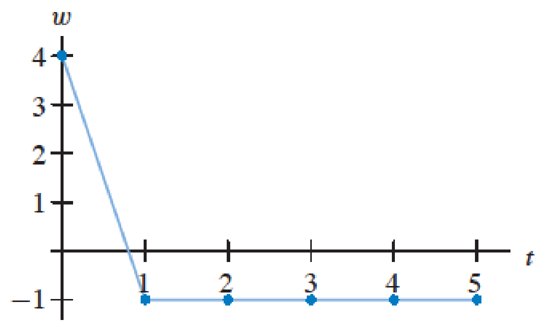
k	t_k	y_k	m_k
0	0	1	0.84
1	0.5	1.42	0.99
2	1.0	1.91	0.94
3	1.5	2.38	0.68
4	2.0	2.73	0.40
5	2.5	2.93	0.21
6	3.0	3.03	



5.

Table 1.5
Results of Euler's method

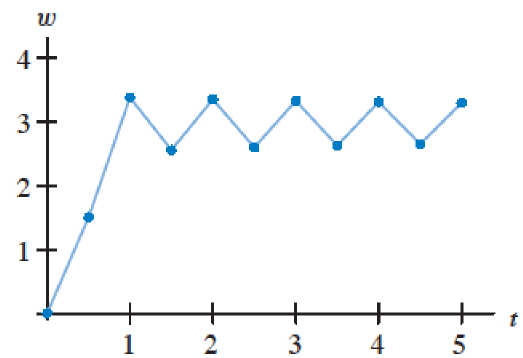
k	t_k	w_k	m_k
0	0	4	−5
1	1	−1	0
2	2	−1	0
3	3	−1	0
4	4	−1	0
5	5	−1	



6.

Table 1.6
Results of Euler's method (shown rounded to two decimal places)

k	t_k	w_k	m_k
0	0	0	3
1	0.5	1.5	3.75
2	1.0	3.38	−1.64
3	1.5	2.55	1.58
4	2.0	3.35	−1.50
5	2.5	2.59	1.46
6	3.0	3.32	−1.40
7	3.5	2.62	1.36
8	4.0	3.31	−1.31
9	4.5	2.65	1.28
10	5.0	3.29	



14. Euler's method is not accurate in either case because the step size is too large. In Exercise 5, the approximate solution "jumps onto" an equilibrium solution. In Exercise 6, the approximate solution "crisscrosses" a different equilibrium solution. Approximate solutions generated with smaller values of Δt indicate that the actual solutions do not exhibit this behavior (see the Existence and Uniqueness Theorem of Section 1.5).

15.

Table 1.13
Results of Euler's method with $\Delta t = 1.0$ (shown to two decimal places)

k	t_k	y_k	m_k
0	0	1	1
1	1	2	1.41
2	2	3.41	1.85
3	3	5.26	2.29
4	4	7.56	

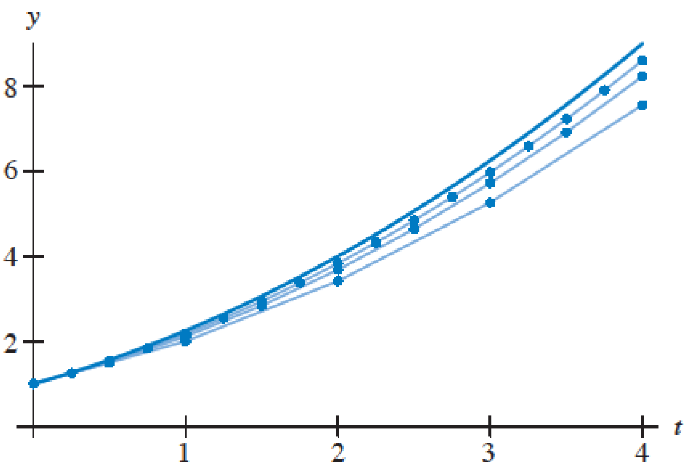
Table 1.14
Results of Euler's method with $\Delta t = 0.5$ (shown to two decimal places)

k	t_k	y_k	m_k	k	t_k	y_k	m_k
0	0	1	1	5	2.5	4.64	2.15
1	0.5	1.5	1.22	6	3.0	5.72	2.39
2	1.0	2.11	1.45	7	3.5	6.91	2.63
3	1.5	2.84	1.68	8	4.0	8.23	
4	2.0	3.68	1.92				

Table 1.15
Results of Euler's method with $\Delta t = 0.25$ (shown to two decimal places)

k	t_k	y_k	m_k	k	t_k	y_k	m_k
0	0	1	1	9	2.25	4.32	2.08
1	0.25	1.25	1.12	10	2.50	4.84	2.20
2	0.50	1.53	1.24	11	2.75	5.39	2.32
3	0.75	1.84	1.36	12	3.0	5.97	2.44
4	1.0	2.18	1.48	13	3.25	6.58	2.56
5	1.25	2.55	1.60	14	3.50	7.23	2.69
6	1.50	2.94	1.72	15	3.75	7.90	2.81
7	1.75	3.37	1.84	16	4.0	8.60	
8	2.0	3.83	1.96				

The slopes in the slope field are positive and increasing. Hence, the graphs of all solutions are concave up. Since Euler's method uses line segments to approximate the graph of the actual solution, the approximate solutions will always be less than the actual solution. This error decreases as the step size decreases.



2. Since $y(0) = 1$ is between the equilibrium solutions $y_2(t) = 0$ and $y_3(t) = 2$, we must have $0 < y(t) < 2$ for all t because the Uniqueness Theorem implies that graphs of solutions cannot cross (or even touch in this case).

4. Because $y_1(0) < y(0) < y_2(0)$, the solution $y(t)$ must satisfy $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Hence $-1 < y(t) < 1 + t^2$ for all t.

9. **(a)** To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

and

$$\begin{aligned} -y_1^2 + y_1 + 2y_1t^2 + 2t - t^2 - t^4 &= -(t^2)^2 + (t^2) + 2(t^2)t^2 + 2t - t^2 - t^4 \\ &= 2t. \end{aligned}$$

To check that $y_2(t) = t^2 + 1$ is a solution, we compute

$$\frac{dy_2}{dt} = 2t$$

and

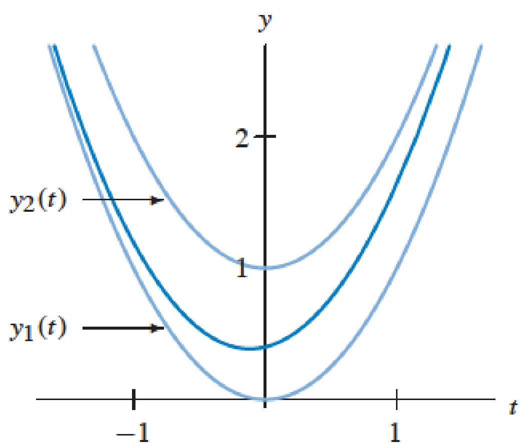
$$\begin{aligned} -y_2^2 + y_2 + 2y_2t^2 + 2t - t^2 - t^4 &= -(t^2+1)^2 + (t^2+1) + 2(t^2+1)t^2 \\ &\quad + 2t - t^2 - t^4 \\ &= 2t. \end{aligned}$$

(b) The initial values of the two solutions are $y_1(0) = 0$ and $y_2(0) = 1$. Thus if $y(t)$ is a solution and $y_1(0) = 0 < y(0) < 1 = y_2(0)$, then we can apply the Uniqueness Theorem to obtain

$$y_1(t) = t^2 < y(t) < t^2 + 1 = y_2(t)$$

for all t. Note that since the differential equation satisfies the hypothesis of the Existence and Uniqueness Theorem over the entire ty-plane, we can continue to extend the solution as long as it does not escape to $\pm\infty$ in finite time. Since it is bounded above and below by solutions that exist for all time, $y(t)$ is defined for all time also.

(c)



10. **(a)** If $y(t) = 0$ for all t, then $dy/dt = 0$ and $2\sqrt{|y(t)|} = 0$ for all t. Hence, the function that is constantly zero satisfies the differential equation.

(b) First, consider the case where $y > 0$. The differential equation reduces to $dy/dt = 2\sqrt{y}$. If we separate variables and integrate, we obtain

$$\sqrt{y} = t - c,$$

where c is any constant. The graph of this equation is the half of the parabola $y = (t - c)^2$ where $t \geq c$.

Next, consider the case where $y < 0$. The differential equation reduces to $dy/dt = 2\sqrt{-y}$. If we separate variables and integrate, we obtain

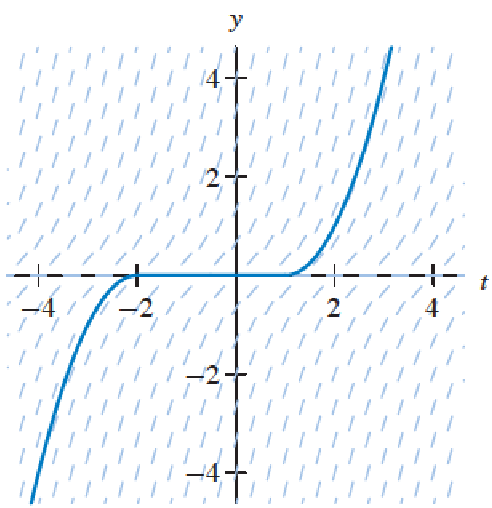
$$\sqrt{-y} = d - t,$$

where d is any constant. The graph of this equation is the half of the parabola $y = -(d - t)^2$ where $t \leq d$.

To obtain all solutions, we observe that any choice of constants c and d where $c \geq d$ leads to a solution of the form

$$y(t) = \begin{cases} -(d-t)^2, & \text{if } t \leq d; \\ 0, & \text{if } d \leq t \leq c; \\ (t-c)^2, & \text{if } t \geq c. \end{cases}$$

(See the following figure for the case where $d = -2$ and $c = 1$.)



(c) The partial derivative $\partial f/\partial y$ of $f(t, y) = \sqrt{|y|}$ does not exist along the t-axis.

(d) If $y_0 = 0$, `HPGSolver` plots the equilibrium solution that is constantly zero. If $y_0 \neq 0$, it plots a solution whose graph crosses the t-axis. This is a solution where $c = d$ in the formula given above.

11. The key observation is that the differential equation is not defined when $t = 0$.

(a) Note that $dy_1/dt = 0$ and $y_1/t^2 = 0$, so $y_1(t)$ is a solution.

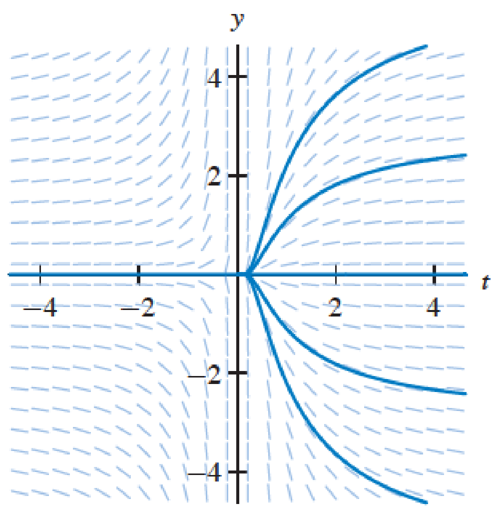
(b) Separating variables, we have

$$\int \frac{dy}{y} = \int \frac{dt}{t^2}.$$

Solving for y we obtain $y(t) = ce^{-1/t}$, where c is any constant. Thus, for any real number c, define the function $y_c(t)$ by

$$y_c(t) = \begin{cases} 0 & \text{for } t \le 0; \\ ce^{-1/t} & \text{for } t > 0. \end{cases}$$

For each c, $y_c(t)$ satisfies the differential equation for all $t \neq 0$.



There are infinitely many solutions of the form $y_c(t)$ that agree with $y_1(t)$ for $t < 0$.

(c) Note that $f(t, y) = y/t^2$ is not defined at $t = 0$. Therefore, we *cannot* apply the Uniqueness Theorem for the initial condition $y(0) = 0$. The "solution" $y_c(t)$ given in part (b) actually represents two solutions, one for $t < 0$ and one for $t > 0$.

18. **(a)** Solving for r, we get

$$r = \left(\frac{3v}{4\pi}\right)^{1/3}.$$

Consequently,

$$\begin{aligned} s(t) &= 4\pi\left(\frac{3v}{4\pi}\right)^{2/3} \\ &= cv(t)^{2/3}, \end{aligned}$$

where c is a constant. Since we are assuming that the rate of growth of $v(t)$ is proportional to its surface area $s(t)$, we have

$$\frac{dv}{dt} = kv^{2/3},$$

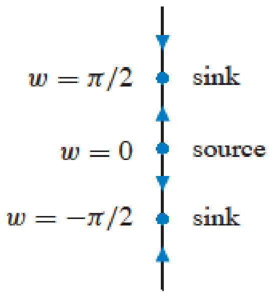
where k is a constant.

(b) The partial derivative with respect to v of dv/dt does not exist at $v = 0$. Hence the Uniqueness Theorem tells us nothing about the uniqueness of solutions that involve $v = 0$. In fact, if we use the techniques described in the section related to the uniqueness of solutions for $dy/dt = 3y^{2/3}$, we can find infinitely many solutions with this initial condition.

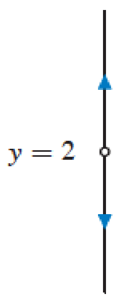
(c) Since it does not make sense to talk about rain drops with negative volume, we always have $v \geq 0$. Once $v > 0$, the evolution of the drop is completely determined by the differential equation.

What is the physical significance of a drop with $v = 0$? It is tempting to interpret the fact that solutions can have $v = 0$ for an arbitrary amount of time before beginning to grow as a statement that the rain drops can spontaneously begin to grow at any time. Since the model gives no information about when a solution with $v = 0$ starts to grow, it is not very useful for the understanding the initial formation of rain drops. The safest assertion is to say is the model breaks down if $v = 0$.

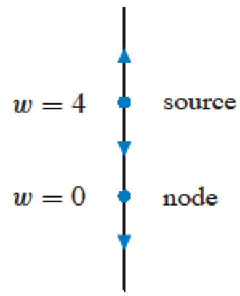
4. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = w \cos w$, the equilibrium points are $w = 0$ and $w = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $w \cos w$ alternates positive and negative at successive zeros. It is negative for $-\pi/2 < w < 0$ and positive for $0 < w < \pi/2$. Therefore, $w = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



6. This equation has no equilibrium points, but the equation is not defined at $y = 2$. For $y > 2$, $dy/dt > 0$, so solutions increase. If $y < 2$, $dy/dt < 0$, so solutions decrease. The solutions approach the point $y = 2$ as time decreases and actually arrive there in finite time.



8. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = 3w^3 - 12w^2$, the equilibrium points are $w = 0$ and $w = 4$. Since $f(w) < 0$ for $w < 0$ and $0 < w < 4$, and $f(w) > 0$ for $w > 4$, the equilibrium point at $w = 0$ is a node and the equilibrium point at $w = 4$ is a source.



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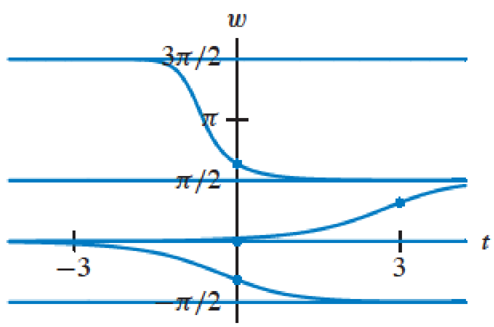

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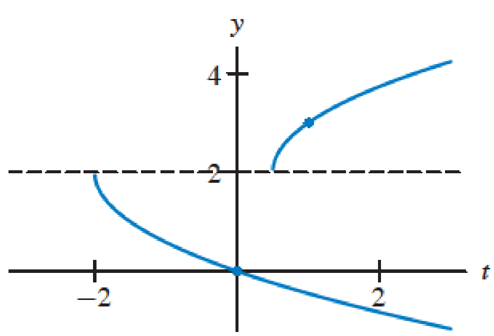

The equation is undefined at $y = 2$.

20.