2. Note that $dy/dt = 0$ for all t only if $y^2 - 2 = 0$. Therefore, the only equilibrium solutions are $y(t) = -\sqrt{2}$ for all t and $y(t) = +\sqrt{2}$ for all t.

4. **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$, $P = 50$, or $P = 200$.

(b) The population is increasing if $dP/dt > 0$. That is, $P < 0$ or $50 < P < 200$. Note, $P < 0$ might be considered "nonphysical" for a population model.

(c) The population is decreasing if $dP/dt < 0$. That is, $0 < P < 50$ or $P > 200$.

5. In order to answer the question, we first need to analyze the sign of the polynomial $y^3 - y^2 - 12y$. Factoring, we obtain

$$y^3 - y^2 - 12y = y(y^2 - y - 12) = y(y-4)(y+3).$$

(a) The equilibrium solutions correspond to the values of y for which $dy/dt = 0$ for all t. For this equation, $dy/dt = 0$ for all t if $y = -3$, $y = 0$, or $y = 4$.

(b) The solution $y(t)$ is increasing if $dy/dt > 0$. That is, $-3 < y < 0$ or $y > 4$.

(c) The solution $y(t)$ is decreasing if $dy/dt < 0$. That is, $y < -3$ or $0 < y < 4$.

6. **(a)** The rate of change of the amount of radioactive material is dr/dt. This rate is proportional to the amount r of material present at time t. With $-\lambda$ as the proportionality constant, we obtain the differential equation

$$\frac{dr}{dt} = -\lambda r.$$

Note that the minus sign (along with the assumption that λ is positive) means that the material decays.

(b) The only additional assumption is the initial condition $r(0) = r_0$. Consequently, the corresponding initial-value problem is

$$\frac{dr}{dt} = -\lambda r, \quad r(0) = r_0.$$

11. The solution of $dR/dt = kR$ with $R(0) = 4{,}000$ is

$$R(t) = 4{,}000\, e^{kt}.$$

Setting $t = 6$, we have $R(6) = 4{,}000\, e^{(k)(6)} = 130{,}000$. Solving for k, we obtain

$$k = \tfrac{1}{6} \ln\left(\tfrac{130{,}000}{4{,}000}\right) \approx 0.58.$$

Therefore, the rabbit population in the year 2010 would be $R(10) = 4{,}000\, e^{(0.58 \cdot 10)} \approx 1{,}321{,}198$ rabbits.

12. **(a)** In this analysis, we consider only the case where v is positive. The right-hand side of the differential equation is a quadratic in v, and it is zero if $v = \sqrt{mg/k}$. Consequently, the solution $v(t) = \sqrt{mg/k}$ for all t is an equilibrium solution. If $0 \leq v < \sqrt{mg/k}$, then $dv/dt > 0$, and consequently, $v(t)$ is an increasing function. If $v > \sqrt{mg/k}$, then $dv/dt < 0$, and $v(t)$ is a decreasing function. In either case, $v(t) \to \sqrt{mg/k}$ as $t \to \infty$.

(b) See part (a).

17. Let $P(t)$ be the population at time t, k be the growth-rate parameter, and N be the carrying capacity. The modified models are

(a) $dP/dt = k(1 - P/N)P - 100$

(b) $dP/dt = k(1 - P/N)P - P/3$

(c) $dP/dt = k(1 - P/N)P - a\sqrt{P}$, where a is a positive parameter.

18. **(a)** The differential equation is $dP/dt = 0.3P(1 - P/2500) - 100$. The equilibrium solutions of this equation correspond to the values of P for which $dP/dt = 0$ for all t. Using the quadratic formula, we obtain two such values, $P_1 \approx 396$ and $P_2 \approx 2104$. If $P > P_2$, $dP/dt < 0$, so $P(t)$ is decreasing. If $P_1 < P < P_2$, $dP/dt > 0$, so $P(t)$ is increasing. Hence the solution that satisfies the initial condition $P(0) = 2500$ decreases toward the equilibrium $P_2 \approx 2104$.

(b) The differential equation is $dP/dt = 0.3P(1 - P/2500) - P/3$. The equilibrium solutions of this equation are $P_1 \approx -277$ and $P_2 = 0$. If $P > 0$, $dP/dt < 0$, so $P(t)$ is decreasing. Hence, for $P(0) = 2500$, the population decreases toward $P = 0$ (extinction).

21. **(a)** The term governing the effect of the interaction of x and y on the rate of change of x is $+\beta xy$. Since this term is positive, the presence of y's helps the x population grow. Hence, x is the predator. Similarly, the term $-\delta xy$ in the dy/dt equation implies that when $x > 0$, y's grow more slowly, so y is the prey. If $y = 0$, then $dx/dt < 0$, so the predators will die out; thus, they must have insufficient alternative food sources. The prey has no limits on its growth other than the predator since, if $x = 0$, then $dy/dt > 0$ and the population increases exponentially.

(b) Since $-\beta xy$ is negative and $+\delta xy$ is positive, x suffers due to its interaction with y and y benefits from its interaction with x. Hence, x is the prey and y is the predator. The predator has other sources of food than the prey since $dy/dt > 0$ even if $x = 0$. Also, the prey has a limit on its growth due to the $-\alpha x^2/N$ term.

22. **(a)** We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what effect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what effect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Paul is wrong.

(b) At first glance, they should have seen the equilibrium solution $y(t) = -1$ for all t because $dy/dt = 0$ for any constant function and $y = -1$ implies that

$$\frac{y+1}{t+1} = 0$$

independent of t.

Strictly speaking the differential equation is not defined for $t = -1$, and hence the solutions are not defined for $t = -1$.

3. In order to find one such $f(t, y)$, we compute the derivative of $y(t)$. We obtain

$$\frac{dy}{dt} = \frac{de^{t^3}}{dt} = 3t^2 e^{t^3}.$$

Now we replace e^{t^3} in the last expression by y and get the differential equation

$$\frac{dy}{dt} = 3t^2 y.$$

8. Separating variables and integrating, we obtain

$$\int \frac{1}{2-y}\, dy = \int dt$$

$$-\ln|2-y| = t + c$$

$$\ln|2-y| = -t + c_1,$$

where we have replaced $-c$ with c_1. Then

$$|2-y| = k_1 e^{-t},$$

where $k_1 = e_1^c$. We can drop the absolute value signs if we replace $\pm k_1$ with k_2, that is, if we allow k_2 to be either positive or negative. Then we have

$$2 - y = k_2 e^{-t}$$

$$y = 2 - k_2 e^{-t}.$$

This could also be written as $y(t) = ke^{-t} + 2$, where we replace $-k_2$ with k. Note that $k = 0$ gives the equilibrium solution.

9. We separate variables and integrate to obtain

$$\int e^y\, dy = \int dt$$

$$e^y = t + c,$$

where c is any constant. We obtain $y(t) = \ln(t + c)$.

12. Separating variables and integrating, we obtain

$$\int y\,dy = \int t\,dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + k$$

$$y^2 = t^2 + c,$$

where $c = 2k$. Hence,

$$y(t) = \pm\sqrt{t^2 + c},$$

where the initial condition determines the choice of sign.

16. Note that there is an equilibrium solution of the form $y = -1/2$.

Separating variables and integrating, we have

$$\int \frac{1}{2y+1}\,dy = \int \frac{1}{t}\,dt$$

$$\frac{1}{2}\ln|2y+1| = \ln|t| + c$$

$$\ln|2y+1| = (\ln t^2) + c$$

$$|2y+1| = c_1 t^2,$$

where $c_1 = e^c$. We can eliminate the absolute value signs by allowing the constant c_1 to be either positive or negative. In other words, $2y + 1 = k_1 t^2$, where $k_1 = \pm c_1$. Hence,

$$y(t) = kt^2 - \tfrac{1}{2},$$

where $k = k_1/2$, or $y(t)$ is the equilibrium solution with $y = -1/2$.

17. First of all, the equilibrium solutions are $y = 0$ and $y = 1$. Now suppose $y \neq 0$ and $y \neq 1$. We separate variables to obtain

$$\int \frac{1}{y(1-y)}\,dy = \int dt = t + c,$$

where c is any constant. To integrate, we use partial fractions. Write

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

We must have $A = 1$ and $-A + B = 0$. Hence, $A = B = 1$ and

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Consequently,

$$\int \frac{1}{y(1-y)}\,dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|.$$

After integration, we have

$$\ln\left|\frac{y}{1-y}\right| = t + c$$

$$\left|\frac{y}{1-y}\right| = c_1 e^t,$$

where $c_1 = e^c$ is any positive constant. To remove the absolute value signs, we replace the positive constant c_1 with a constant k that can be any real number and get

$$y(t) = \frac{ke^t}{1 + ke^t},$$

where $k = \pm c_1$. If $k = 0$, we get the first equilibrium solution. The formula $y(t) = ke^t/(1 + ke^t)$ yields all the solutions to the differential equation except for the equilibrium solution $y(t) = 1$.

25. Separating variables and integrating, we have

$$\int \frac{1}{x}\,dx = \int -t\,dt$$

$$\ln|x| = -\frac{t^2}{2} + c$$

$$|x| = k_1 e^{-t^2/2},$$

where $k_1 = e^c$. We can eliminate the absolute value signs by allowing the constant k_1 to be either positive or negative. Thus, the general solution is

$$x(t) = ke^{-t^2/2}$$

where $k = \pm k_1$. Using the initial condition to solve for k, we have

$$\frac{1}{\sqrt{\pi}} = x(0) = ke^0 = k.$$

Therefore,

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}.$$

26. Separating variables and integrating, we have

$$\int \frac{1}{y}\,dy = \int t\,dt$$

$$\ln|y| = \frac{t^2}{2} + c$$

$$|y| = k_1 e^{t^2/2},$$

where $k_1 = e^c$. We can eliminate the absolute value signs by allowing the constant k_1 to be either positive or negative. Thus, the general solution can be written as

$$y(t) = ke^{t^2/2}.$$

Using the initial condition to solve for k, we have

$$3 = y(0) = ke^0 = k.$$

Therefore, $y(t) = 3e^{t^2/2}$.

30. Rewriting the equation as

$$\frac{dy}{dt} = \frac{t}{(1-t^2)y},$$

we separate variables and integrate obtaining

$$\int y\,dy = \int \frac{t}{1-t^2}\,dt$$

$$\frac{y^2}{2} = -\frac{1}{2}\ln|1-t^2| + c$$

$$y = \pm\sqrt{-\ln|1-t^2| + k}.$$

Since $y(0) = 4$ is positive, we use the positive square root and solve

$$4 = y(0) = \sqrt{-\ln|1| + k} = \sqrt{k}$$

for k. We obtain $k = 16$. Hence,

$$y(t) = \sqrt{16 - \ln(1-t^2)}.$$

We may replace $|1-t^2|$ with $(1-t^2)$ because the solution is only defined for $-1 < t < 1$.

32. First we find the general solution by writing the differential equation as

$$\frac{dy}{dt} = (t+2)y^2,$$

separating variables, and integrating. We have

$$\int \frac{1}{y^2}\,dy = \int (t+2)\,dt$$

$$-\frac{1}{y} = \frac{t^2}{2} + 2t + c$$

$$= \frac{t^2 + 4t + c_1}{2},$$

where $c_1 = 2c$. Inverting and multiplying by -1 produces

$$y(t) = \frac{-2}{t^2 + 4t + c_1}.$$

Setting

$$1 = y(0) = \frac{-2}{c_1}$$

and solving for c_1, we obtain $c_1 = -2$. So

$$y(t) = \frac{-2}{t^2 + 4t - 2}.$$

35. We separate variables to obtain

$$\int \frac{dy}{1+y^2} = \int t\,dt$$

$$\arctan y = \frac{t^2}{2} + c,$$

where c is a constant. Hence the general solution is

$$y(t) = \tan\left(\frac{t^2}{2} + c\right).$$

Next we find c so that $y(0) = 1$. Solving

$$1 = \tan\left(\frac{0^2}{2} + c\right)$$

yields $c = \pi/4$, and the solution to the initial-value problem is

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right).$$

41. **(a)** If we let k denote the proportionality constant in Newton's law of cooling, the differential equation satisfied by the temperature T of the chocolate is

$$\frac{dT}{dt} = k(T - 70).$$

We also know that $T(0) = 170$ and that $dT/dt = -20$ at $t = 0$. Therefore, we obtain k by evaluating the differential equation at $t = 0$. We have

$$-20 = k(170 - 70),$$

so $k = -0.2$. The initial-value problem is

$$\frac{dT}{dt} = -0.2(T - 70), \quad T(0) = 170.$$

(b) We can solve the initial-value problem in part (a) by separating variables. We have

$$\int \frac{dT}{T - 70} = \int -0.2\, dt$$

$$\ln|T - 70| = -0.2t + k$$

$$|T - 70| = ce^{-0.2t}.$$

Since the temperature of the chocolate cannot become lower than the temperature of the room, we can ignore the absolute value and conclude

$$T(t) = 70 + ce^{-0.2t}.$$

Now we use the initial condition $T(0) = 170$ to find the constant c because

$$170 = T(0) = 70 + ce^{-0.2(0)},$$

which implies that $c = 100$. The solution is

$$T = 70 + 100e^{-0.2t}.$$

In order to find t so that the temperature is 110° F, we solve

$$110 = 70 + 100e^{-0.2t}$$

for t obtaining

$$\tfrac{2}{5} = e^{-0.2t}$$

$$\ln\tfrac{2}{5} = -0.2t$$

so that

$$t = \frac{\ln(2/5)}{-0.2} \approx 4.6.$$