4. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= ke^{-t} - 4ke^{-t} + 13ke^{-t} \\ &= 10ke^{-t}. \end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/10$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t + \tfrac{1}{10} e^{-t}.$$

7. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 - 5s + 4,$$

so the eigenvalues are $s = 1$ and $s = 4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^t + k_2 e^{4t}.$$

To find a particular solution of the forced equation, a reasonable looking guess is $y_p(t) = ke^{4t}$. However, this guess is a solution of the homogeneous equation, so it is doomed to fail. We make the standard second guess of $y_p(t) = kte^{4t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} - 5\frac{dy_p}{dt} + 4y_p &= (8ke^{4t} + 16kte^{4t}) - 5(ke^{4t} + 4kte^{4t}) + 4kte^{4t} \\ &= 3ke^{4t}. \end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^t + k_2 e^{4t} + \tfrac{1}{3} te^{4t}.$$

10. First we derive the general solution. The characteristic polynomial is

$$s^2 + 7s + 12,$$

so the eigenvalues are $s = -3$ and $s = -4$. To find the general solution of the forced equation, we also need a particular solution. We guess $y_p(t) = ke^{-t}$ and find that $y_p(t)$ is a solution only if $k = 1/2$. Therefore, the general solution is

$$y(t) = k_1 e^{-3t} + k_2 e^{-4t} + \tfrac{1}{2} e^{-t}.$$

To find the solution with the initial conditions $y(0) = 2$ and $y'(0) = 1$, we compute

$$y'(t) = -3k_1 e^{-3t} - 4k_2 e^{-4t} - \tfrac{1}{2} e^{-t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{2} = 2 \\ -3k_1 - 4k_2 - \frac{1}{2} = 1. \end{cases}$$

Solving, we have $k_1 = 15/2$ and $k_2 = -6$, so the solution of the initial-value problem is

$$y(t) = \tfrac{15}{2} e^{-3t} - 6e^{-4t} + \tfrac{1}{2} e^{-t}.$$

18. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 20.$$

So the eigenvalues are $s = -2 \pm 4i$, and the general solution of the unforced equation is

$$k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-4t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 20y_p &= 16ke^{-4t} - 16ke^{-4t} + 20ke^{-4t} \\ &= 20ke^{-4t}. \end{aligned}$$

So $k = 1/20$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t + \tfrac{1}{20} e^{-4t}.$$

(b) The derivative of the general solution is

$$\begin{aligned} y'(t) = {} & -k_1 e^{-2t} \cos 4t - 4k_1 e^{-2t} \sin 4t \\ & -2k_2 e^{-2t} \sin 4t + 4k_2 e^{-2t} \cos 4t - \tfrac{1}{5} e^{-4t}. \end{aligned}$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{20} = 0 \\ -2k_1 + 4k_2 - \frac{1}{5} = 0. \end{cases}$$

Solving, we find that $k_1 = -1/20$ and $k_2 = 1/40$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{20} e^{-2t} \cos 4t + \tfrac{1}{40} e^{-2t} \sin 4t + \tfrac{1}{20} e^{-4t}.$$

(c) From the formula for the general solution, we see that every solution tends to zero. The e^{-4t} term in the general solution tends to zero quickest, so for large t, the solution is very close to the unforced solution. All solutions tend to zero and all but the purely exponential one oscillates with frequency $2/\pi$ and an amplitude that decreases at the rate of e^{-2t}.

35. **(a)** The general solution of the homogeneous equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution to the nonhomogeneous equation, we guess

$$y_p(t) = at^2 + bt + c.$$

Substituting $y_p(t)$ into the differential equation yields

$$\begin{aligned} \frac{d^2y_p}{dt^2} + 4y_p &= t - \frac{t}{20} \\ 2a + 4(at^2 + bt + c) &= t - \frac{t}{20} \\ (4a)t^2 + (4b)t + (2a + 4c) &= t - \frac{t}{20}. \end{aligned}$$

Equating coefficients, we obtain the simultaneous equations

$$\left\{ \begin{aligned} 4a &= -\tfrac{1}{20} \\ 4b &= 1 \\ 2a + 4c &= 0. \end{aligned} \right.$$

Therefore, $a = -1/80$, $b = 1/4$, and $c = 1/160$ yield a solution to the nonhomogeneous equation, and the general solution of the nonhomogeneous equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t - \tfrac{1}{80}t^2 + \tfrac{1}{4}t + \tfrac{1}{160}.$$

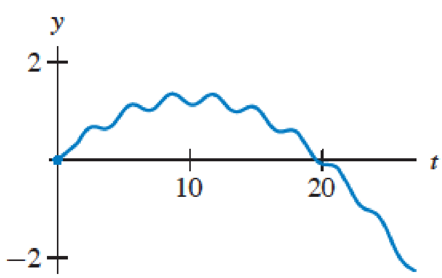
(b) To solve the initial-value problem with $y(0) = 0$ and $y'(0) = 0$, we have

$$\left\{ \begin{aligned} k_1 + \tfrac{1}{160} &= 0 \\ 2k_2 + \tfrac{1}{4} &= 0. \end{aligned} \right.$$

Therefore, $k_1 = -1/160$ and $k_2 = -1/8$, and the solution is

$$y(t) = -\tfrac{1}{160} \cos 2t - \tfrac{1}{8} \sin 2t - \tfrac{1}{80}t^2 + \tfrac{1}{4}t + \tfrac{1}{160}.$$

(c) Since the solution to the homogeneous equation is periodic with a small amplitude and since the solution to the nonhomogeneous equation goes to $-\infty$ at a rate determined by $-t^2/80$, the solution tends to $-\infty$.



42. **(a)** To find the general solution of the unforced equation, we note that the characteristic polynomial is s^2+4, which has roots $s=\pm 2i$. So the general solution of the unforced equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution of the forced equation we guess

$$y_p(t) = a + bt + ct^2 + de^t.$$

Substituting this guess into the differential equation yields

$$(2c + de^t) + 4(a + bt + ct^2 + de^t) = 6 + t^2 + e^t,$$

which simplifies to

$$(2c + 4a) + (4b)t + (4c)t^2 + (5d)e^t = 6 + t^2 + e^t.$$

So $d = 1/5$, $c = 1/4$, $b = 0$, and $a = 1/8$ yield a solution, and the general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(b) Note that

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t + \tfrac{1}{2}t + \tfrac{1}{5}e^t.$$

To obtain the desired initial conditions we must solve

$$\begin{cases} k_1 + \frac{11}{8} + \frac{1}{5} = 0 \\ \quad 2k_2 + \frac{1}{5} = 0, \end{cases}$$

which yields $k_1 = -63/40$ and $k_2 = -1/10$. The solution of the initial-value problem is

$$y(t) = -\tfrac{63}{40} \cos 2t - \tfrac{1}{10} \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(c) This solution tends to infinity at a rate that is determined by $e^t/5$ because this term dominates when t is large.

1. Since the equilibrium point is at the origin and the system has only polynomial terms, the linearized system is just the linear terms in dx/dt and dy/dt, that is,

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = -2y.$$

2. From the linearized system in Exercise 1, we see (without any calculation) that the eigenvalues are 1 and -2. Hence, the origin is a saddle.

3. The Jacobian matrix for this system is

$$\begin{pmatrix} 2x + 3\cos 3x & 0 \\ -y\cos xy & 2 - x\cos xy \end{pmatrix},$$

and evaluating at $(0, 0)$, we get

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

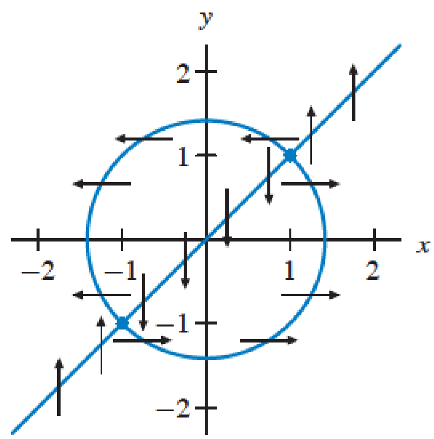
So the linearized system at the origin is

$$\frac{dx}{dt} = 3x$$
$$\frac{dy}{dt} = 2y.$$

4. From the linearized system in Exercise 3, we see (without any calculation) that the eigenvalues are 3 and 2. Hence, the origin is a source.

5. The x-nullcline is where $dx/dt = 0$, that is, the line $y = x$. The y-nullcline is where $dy/dt = 0$, that is, the circle $x^2 + y^2 = 2$.

Along the x-nullcline, $dy/dt < 0$ if and only if $-\sqrt{2} < x < \sqrt{2}$. Along the y-nullcline, $dx/dt < 0$ if and only if $y > x$.



6. This system is not a Hamiltonian system. If it were, then we would have

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial x} = \frac{dy}{dt}$$

for some function $H(x, y)$. In that case, equality of mixed partials would imply that

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right).$$

For this system, we have

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = 2y \quad \text{and} \quad -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) = -2y.$$

Since these two partials do not agree, no such function $H(x, y)$ exists.

11. True. The x-nullcline is where $dx/dt = 0$ and the y-nullcline is where $dy/dt = 0$, so any point in common must be an equilibrium point.

14. False. The Jacobian matrix at an equilibrium point (x_0, y_0) is

$$\begin{pmatrix} f'(x_0) & 0 \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix},$$

so its eigenvalues are $f'(x_0)$ and

$$\frac{\partial g}{\partial y}(x_0, y_0).$$

Since this partial derivative could be positive, negative, or zero, the equilibrium point could be a source, a saddle, or one of the zero eigenvalue types.

15. **(a)** Setting $dx/dt = 0$ and $dy/dt = 0$, we obtain the simultaneous equations

$$\begin{cases} \quad x - 3y^2 = 0 \\ x - 3y - 6 = 0. \end{cases}$$

Solving for x and y yields the equilibrium points $(12, 2)$ and $(3, -1)$.

To determine the type of an equilibrium point, we compute the Jacobian matrix. We get

$$\begin{pmatrix} 1 & -6y \\ 1 & -3 \end{pmatrix}.$$

At $(12, 2)$, the Jacobian is

$$\begin{pmatrix} 1 & -12 \\ 1 & -3 \end{pmatrix},$$

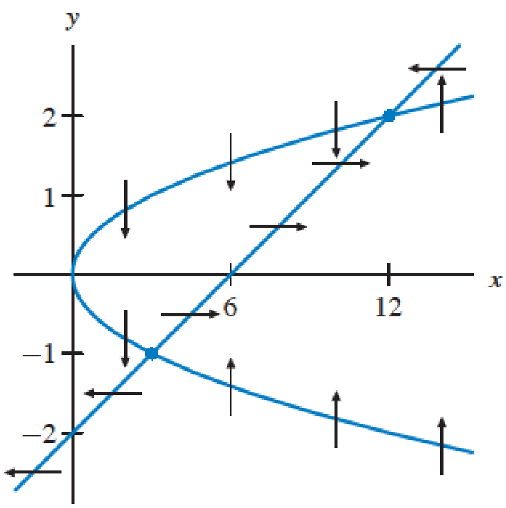
and its eigenvalues are $-1 \pm 2\sqrt{2}\,i$. Hence, $(12, 2)$ is a spriral sink.

At $(3, -1)$, the Jacobian matrix is

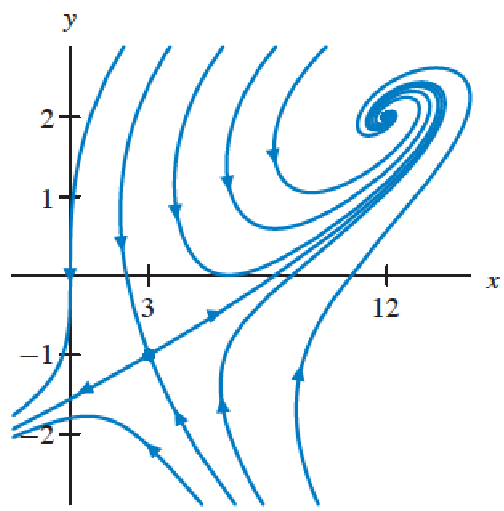
$$\begin{pmatrix} 1 & 6 \\ 1 & -3 \end{pmatrix},$$

and the eigenvalues are $-1 \pm \sqrt{10}$. So $(3, -1)$ is a saddle.

(b) The x-nullcline is $x = 3y^2$, and the y-nullcline is $x = 3y + 6$. We compute the direction of the vector field by computing the sign of dy/dt on the x-nullcline and the sign of dx/dt on the y-nullcline.



(c)



17. **(a)** To find the equilibria, we solve the system of equations

$$\begin{cases} 4x - x^2 - xy = 0 \\ 6y - 2y^2 - xy = 0. \end{cases}$$

We obtain the four equilibrium points $(0, 0)$, $(0, 3)$, $(4, 0)$, and $(2, 2)$.

To classify the equilibria, we compute the Jacobian

$$\begin{pmatrix} 4 - 2x - y & -x \\ -y & 6 - 4y - x \end{pmatrix},$$

evaluate at each equilibrium point, and compute the eigenvalues.

At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

The eigenvalues are 4 and 6, so the origin is a source.

At $(0, 3)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ -3 & -6 \end{pmatrix}.$$

The eigenvalues are 1 and -6, so $(0, 3)$ is a saddle.

At $(4, 0)$, the Jacobian matrix is

$$\begin{pmatrix} -4 & -4 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are -4 and 2, so $(4, 0)$ is a saddle.

Finally, at $(2, 2)$ the Jacobian matrix is

$$\begin{pmatrix} -6 & -2 \\ -2 & -4 \end{pmatrix}.$$

The eigenvalues are $-5 \pm \sqrt{5}$. Both are negative, so $(2, 2)$ is a sink.

(b) The x-nullcline satisfies the equation $4x - x^2 - xy = 0$, which can be rewritten as

$$x(4 - x - y) = 0.$$

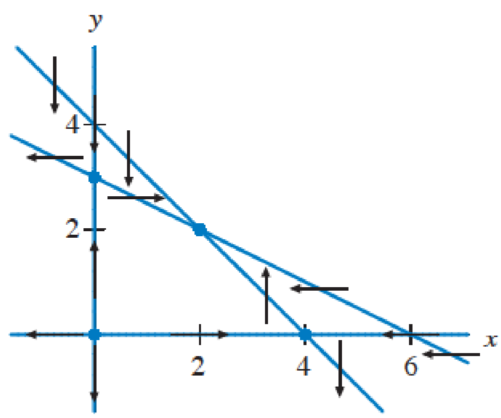
We get two lines, $x = 0$ (the y-axis) and $y = 4 - x$.

The y-nullcline satisfies the equation $6y - 2y^2 - xy = 0$, which can be rewritten as

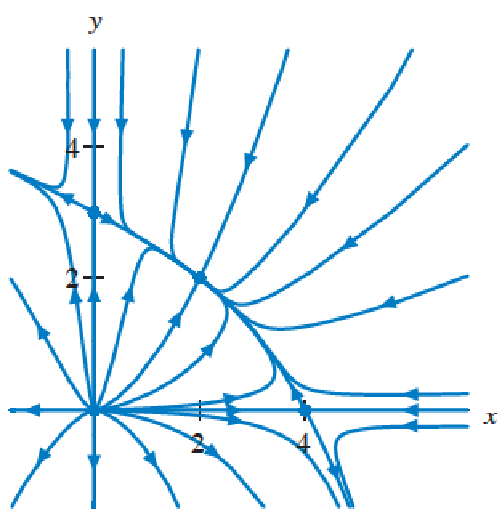
$$y(6 - 2y - x) = 0.$$

We get two lines, $y = 0$ (the x-axis) and $x + 2y = 6$.

We compute the direction of the vector field by computing the sign of dy/dt on the x-nullcline and the sign of dx/dt on the y-nullcline.



(c) The phase portrait is



25. **(a)** The equilibrium points are the solutions of

$$\begin{cases} y^2 - x^2 - 1 = 0 \\ 2xy = 0, \end{cases}$$

that is, $(0, \pm 1)$.

The Jacobian matrix is

$$\begin{pmatrix} -2x & 2y \\ 2y & 2x \end{pmatrix}.$$

At $(0, 1)$, the Jacobian is

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 4$, so its eigenvalues are $\lambda = \pm 2$. The equilibrium point is a saddle.

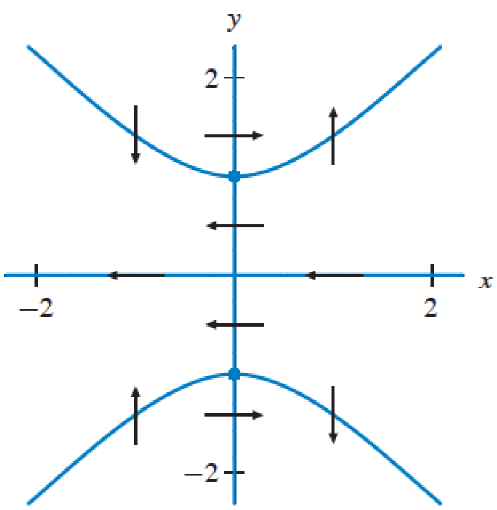
At $(0, -1)$, the Jacobian is

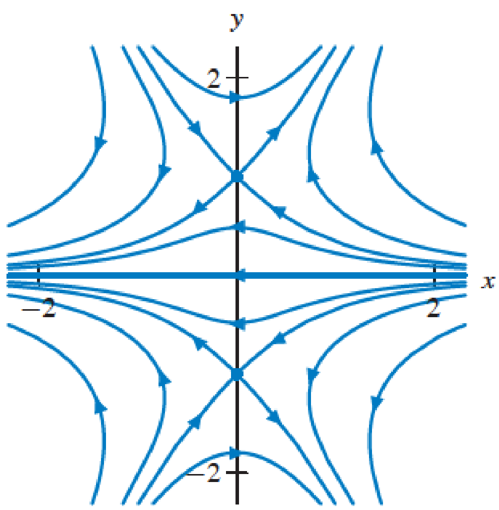
$$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 4$, so its eigenvalues are $\lambda = \pm 2$. The equilibrium point is a saddle.

(b) The x-nullcline is the hyperbola $y^2 - x^2 = 1$, and the y-nullclines are the x- and y-axes.

In the following figures, the nullclines are on the left and the phase portrait is on the right.





(c) To see if the system is Hamiltonian, we compute

$$\frac{\partial(y^2 - x^2 - 1)}{\partial x} = -2x \quad \text{and} \quad -\frac{\partial(2xy)}{\partial y} = -2x.$$

Since these partials agree, the system is Hamiltonian.

The Hamiltonian is a function $H(x, y)$ such that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = y^2 - x^2 - 1 \quad \text{and} \quad \frac{\partial H}{\partial x} = -\frac{dy}{dt} = -2xy.$$

We integrate the second equation with respect to x to see that

$$H(x, y) = -x^2 y + \phi(y),$$

where $\phi(y)$ represents the terms whose derivative with respect to x are zero. Using this expression for $H(x, y)$ in the first equation, we obtain

$$-x^2 + \phi'(y) = y^2 - x^2 - 1.$$

Hence, $\phi'(y) = y^2 - 1$, and we can take $\phi(y) = \frac{1}{3}y^3 - y$. The function

$$H(x, y) = -x^2 y + \frac{y^3}{3} - y$$

is a Hamiltonian function for this system.

(d) To see if the system is a gradient system, we compute

$$\frac{\partial(y^2 - x^2 - 1)}{\partial y} = 2y \quad \text{and} \quad \frac{\partial(2xy)}{\partial x} = 2y.$$

Since these partials agree, the system is a gradient system.

We must now find a function $G(x, y)$ such that

$$\frac{\partial G}{\partial x} = \frac{dx}{dt} = y^2 - x^2 - 1 \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{dy}{dt} = 2xy.$$

Integrating the second equation with respect to y, we obtain

$$G(x, y) = xy^2 + h(x),$$

where $h(x)$ represents the terms whose derivative with respect to y are zero.

Using this expression for $G(x, y)$ in the first equation, we obtain

$$y^2 + h'(x) = y^2 - x^2 - 1.$$

Hence, $h'(x) = -x^2 - 1$, and we can take $h(x) = -\frac{1}{3}x^3 - x$. The function

$$G(x, y) = xy^2 - \frac{x^3}{3} - x$$

is the required function.

27. To see if the system is Hamiltonian, we compute

$$\frac{\partial(-3x+10y)}{\partial x} = -3 \quad \text{and} \quad -\frac{\partial(-x+3y)}{\partial y} = -3.$$

Since these partials agree, the system is Hamiltonian.

To find the Hamiltonian function, we use the fact that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = -3x + 10y.$$

Integrating with respect to y gives

$$H(x, y) = -3xy + 5y^2 + \phi(x),$$

where $\phi(x)$ represents the terms whose derivative with respect to y are zero. Differentiating this expression for $H(x, y)$ with respect to x gives

$$-3y + \phi'(x) = -\frac{dy}{dt} = x - 3y.$$

We choose $\phi(x) = \frac{1}{2}x^2$ and obtain the Hamiltonian function

$$H(x, y) = -3xy + 5y^2 + \frac{x^2}{2}.$$

We know that the solution curves of a Hamiltonian system remain on the level sets of the Hamiltonian function. Hence, solutions of this system satisfy the equation

$$-3xy + 5y^2 + \frac{x^2}{2} = h$$

for some constant h. Multiplying through by 2 yields the equation

$$x^2 - 6xy + 10y^2 = k$$

where $k = 2h$ is a constant.

28. **(a)** To see if the system is Hamiltonian, we compute

$$\frac{\partial(ax+by)}{\partial x}=a \quad \text{and} \quad -\frac{\partial(cx+dy)}{\partial y}=-d.$$

For these partials to agree, we must have $a=-d$.

Assuming that $d=-a$, we want a function $H(x,y)$ such that

$$\frac{\partial H}{\partial y}=\frac{dx}{dt}=ax+by \quad \text{and} \quad \frac{\partial H}{\partial x}=-\frac{dy}{dt}=-cx+ay.$$

We integrate the second equation with respect to x to see that

$$H(x,y)=-\frac{c}{2}x^2+axy+\phi(y),$$

where $\phi(y)$ represents the terms whose derivative with respect to x are zero.

Using this expression for $H(x,y)$ in the first equation, we obtain

$$ax+\phi'(y)=ax+by.$$

In other words, $\phi'(y)=by$, and we can take $\phi(y)=by^2/2$. The function

$$H(x,y)=-\frac{c}{2}x^2+axy+\frac{b}{2}y^2$$

is a Hamiltonian function for this system if $d=-a$.

(b) To see if the system is a gradient system, we compute

$$\frac{\partial(ax+by)}{\partial y}=b \quad \text{and} \quad \frac{\partial(cx+dy)}{\partial x}=c.$$

The linear system is a gradient system if $b=c$.

Assuming that $b=c$, we want a function $G(x,y)$ such that

$$\frac{\partial G}{\partial x}=\frac{dx}{dt}=ax+by \quad \text{and} \quad \frac{\partial G}{\partial y}=\frac{dy}{dt}=bx+dy.$$