1. The simplest solution is an equilibrium solution, and the origin is an equilibrium point for this system. Hence, the equilibrium solution $(x(t), y(t)) = (0, 0)$ for all t is a solution.

2. Note that $dy/dt > 0$ for all (x, y). Hence, there are no equilibrium points for this system.

3. Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2$, and we obtain the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= 1. \end{aligned}$$

5. The equation for dx/dt gives $y = 0$. If $y = 0$, then $\sin(xy) = 0$, so $dy/dt = 0$. Hence, every point on the x-axis is an equilibrium point.

7. First, we check to see if $dx/dt = 2x - 2y^2$ is satisfied. We compute

$$\frac{dx}{dt} = -6e^{-6t} \quad \text{and} \quad 2x - 2y^2 = 2e^{-6t} - 8e^{-6t} = -6e^{-6t}.$$

Second, we check to see if $dy/dt = -3y$. We compute

$$\frac{dy}{dt} = -6e^{-3t} \quad \text{and} \quad -3y = -3(2e^{-3t}) = -6e^{-3t}.$$

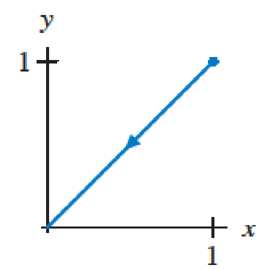
Since both equations are satisfied, $(x(t), y(t))$ is a solution.

9. From the equation for dx/dt, we know that $x(t) = k_1 e^{2t}$, where k_1 is an arbitrary constant, and from the equation for dy/dt, we have $y(t) = k_2 e^{-3t}$, where k_2 is another arbitrary constant. The general solution is $(x(t), y(t)) = (k_1 e^{2t}, k_2 e^{-3t})$.

12. One step of Euler's method is

$$\begin{aligned} (2, 1) + \Delta t\, \mathbf{F}(2, 1) &= (2, 1) + 0.5\,(3, 2) \\ &= (3.5, 2). \end{aligned}$$

13. The point $(1, 1)$ is on the line $y = x$. Along this line, the vector field for the system points toward the origin. Therefore, the solution curve consists of the half-line $y = x$ in the first quadrant. Note that the point $(0, 0)$ is not on this curve.



15. True. First, we check the equation for dx/dt. We have

$$\frac{dx}{dt} = \frac{d(e^{-6t})}{dt} = -6e^{-6t},$$

and

$$2x - 2y^2 = 2(e^{-6t}) - 2(2e^{-3t})^2 = 2e^{-6t} - 8e^{-6t} = -6e^{-6t}.$$

Since that equation holds, we check the equation for dy/dt. We have

$$\frac{dy}{dt} = \frac{d(2e^{-3t})}{dt} = -6e^{-3t},$$

and

$$-3y = -3(2e^{-3t}) = -6e^{-3t}.$$

Since the equations for both dx/dt and dy/dt hold, the function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ is a solution of this system.

16. False. A solution to this system must consist of a pair $(x(t), y(t))$ of functions.

20. True. For an autonomous system, the rates of change of solutions depend only on position, not on time. Hence, if a function $(x_1(t), y_1(t))$ satisfies an autonomous system, then the function given by

$$(x_2(t), y_2(t)) = (x_1(t+T), y_1(t+T)),$$

where T is some constant, satisfies the same system.

23. False. The point $(0, 0)$ is an equilibrium point, so the Uniqueness Theorem guarantees that it is not on the solution curve corresponding to $(1, 0)$.

24. False. From the Uniqueness Theorem, we know that the solution curve with initial condition $(1/2, 0)$ is trapped by other solution curves that it cannot cross (or even touch). Hence, $x(t)$ and $y(t)$ must remain bounded for all t.

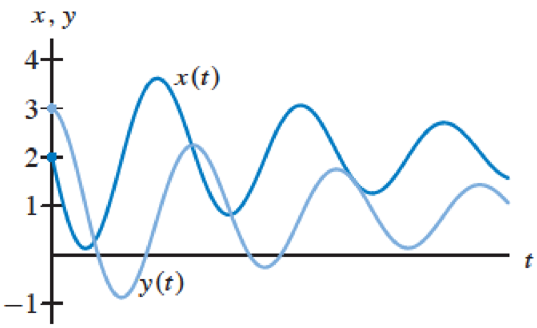
25. False. These solutions are different because they have different values at $t = 0$. However, they do trace out the same curve in the phase plane.

26. True. The solution curve is in the second quadrant and tends toward the equilibrium point $(0, 0)$ as $t \to \infty$. It never touches $(0, 0)$ by the Uniqueness Theorem.

27. False. The function $y(t)$ decreases monotonically, but $x(t)$ increases until it reaches its maximum at $x = -1$. It decreases monotonically after that.

28. False. The graph of $x(t)$ for this solution has exactly one local maximum and no other critical points. The graph of $y(t)$ has four critical points, two local minimums and two local maximums.

31.



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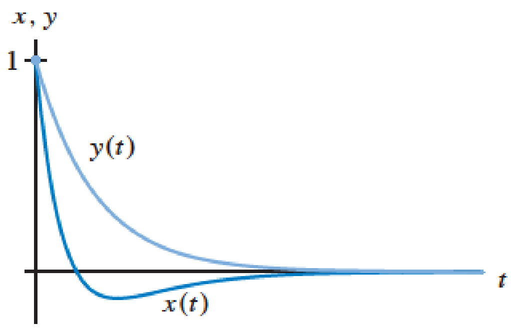

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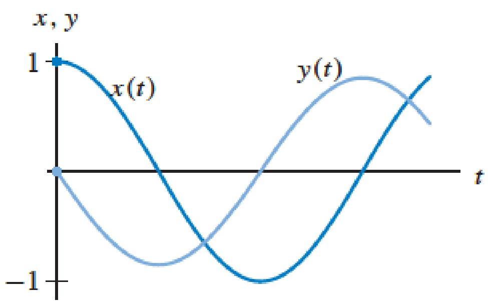

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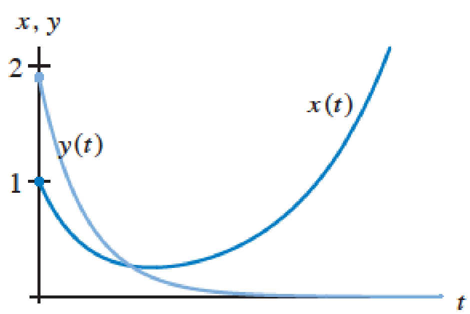


36. **(a)** For this system, we note that the equation for dy/dt depends only on y. In fact, this equation is separable and linear, so we have a choice of techniques for finding the general solution. The general solution for y is $y(t) = -1 + k_1 e^t$, where k_1 can be any constant.

Substituting $y = -1 + k_1 e^t$ into the equation for dx/dt, we have

$$\frac{dx}{dt} = (-1 + k_1 e^t)x.$$

This equation is a homogeneous linear equation, and its general solution is

$$x(t) = k_2 e^{-t + k_1 e^t},$$

where k_2 is any constant. The general solution for the system is therefore

$$(x(t), y(t)) = (k_2 e^{-t + k_1 e^t}, -1 + k_1 e^t),$$

where k_1 and k_2 are constants which we can adjust to satisfy any given initial condition.

(b) Setting $dy/dt = 0$, we obtain $y = -1$. From $dx/dt = xy = 0$, we see that $x = 0$. Therefore, this system has exactly one equilibrium point, $(x, y) = (0, -1)$.

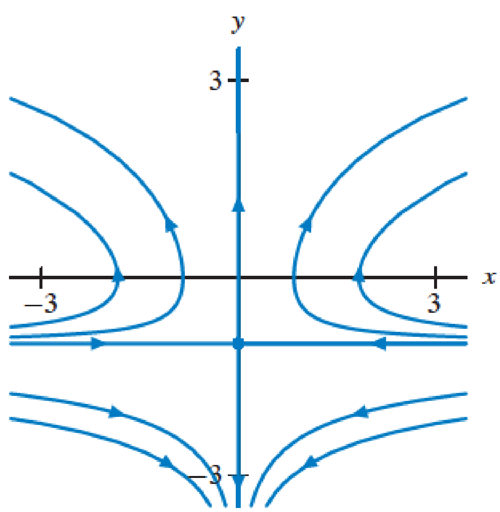
(c) If $(x(0), y(0)) = (1, 0)$, then we must solve the simultaneous equations

$$\begin{cases} k_2 e^{k_1} = 1 \\ -1 + k_1 = 0. \end{cases}$$

Hence, $k_1 = 1$, and $k_2 = 1/e$. The solution to the initial-value problem is

$$(x(t), y(t)) = \left(e^{-1} e^{-t + e^t}, -1 + e^t\right) = \left(e^{e^t - t - 1}, -1 + e^t\right).$$

(d)



1. The characteristic polynomial is $(1 - \lambda)(2 - \lambda)$, so the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

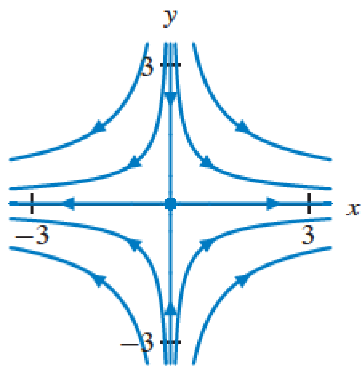
2. The characteristic polynomial is

$$(-\lambda)(-\lambda) - (1)(2) = \lambda^2 - 2,$$

so the eigenvalues are $\lambda = \pm\sqrt{2}$.

3. The system has eigenvalues -2 and 3. One eigenvector associated with $\lambda = 3$ is $(1, 0)$, and one eigenvector associated with $\lambda = -2$ is $(0, 1)$. The general solution is

$$\mathbf{Y}(t) = k_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



4. By definition, the zero vector, $\mathbf{Y}_1$, is never an eigenvector. We can check the others by computing $\mathbf{AY}$. For example,

$$\mathbf{AY}_2 = \mathbf{A} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \mathbf{Y}_2,$$

so $\mathbf{Y}_2$ is an eigenvector (with eigenvalue $\lambda = 1$). On the other hand,

$$\mathbf{AY}_3 = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

which is not a scalar multiple of $\mathbf{Y}_3$, so $\mathbf{Y}_3$ is not an eigenvector. Also, $\mathbf{AY}_4 = 3\mathbf{Y}_4$, so $\mathbf{Y}_4$ is an eigenvector (with eigenvalue $\lambda = 3$). Since we know that $\mathbf{Y}_2$ is an eigenvector and $\mathbf{Y}_5 = -2\mathbf{Y}_2$, $\mathbf{Y}_5$ is also an eigenvector. The vectors $\mathbf{Y}_2$ and $\mathbf{Y}_4$ are two linearly independent eigenvectors corresponding to different eigenvalues. Therefore, $\mathbf{Y}_6$ cannot be an eigenvector because it is neither a scalar multiple of $\mathbf{Y}_2$ nor $\mathbf{Y}_4$.

5. Note that $b \geq 0$ by assumption. The characteristic polynomial is

$$s^2 + bs + 5,$$

so the eigenvalues are

$$s = \frac{-b \pm \sqrt{b^2 - 20}}{2}.$$

If $b > \sqrt{20}$, the harmonic oscillator is overdamped. If $b = \sqrt{20}$, the harmonic oscillator is critically damped. If $0 < b < \sqrt{20}$, the harmonic oscillator is underdamped, and if $b = 0$, the harmonic oscillator is undamped.

7. Every linear system has the origin as an equilibrium point, so the solution to the initial-value problem is the equilibrium solution $\mathbf{Y}(t) = (0, 0)$ for all t.

9. Letting $x(t) = 3\cos 2t$ and $y(t) = \sin 2t$, we have

$$\frac{dx}{dt} = \frac{d(3\cos 2t)}{dt} = -6\sin 2t = -6y$$

and

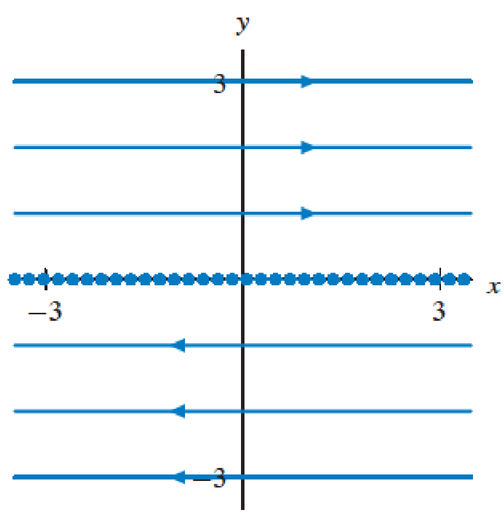
$$\frac{dy}{dt} = \frac{d(\sin 2t)}{dt} = 2\cos t = \frac{2}{3}x.$$

Hence, $\mathbf{Y}(t)$ satisfies the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -6 \\ 2/3 & 0 \end{pmatrix} \mathbf{Y}.$$

10. Written in terms of coordinates, the system is $dx/dt = y$ and $dy/dt = 0$. From the second equation, we see that $y(t) = k_2$, where k_2 is an arbitrary constant. Then $x(t) = k_2 t + k_1$, where k_1 is another arbitrary constant. In vector notation, the general solution is

$$\mathbf{Y}(t) = \begin{pmatrix} k_2 t + k_1 \\ k_2 \end{pmatrix}.$$



11. False. For example, the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$$

has a line of equilibria (the y-axis). Another example is the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Y}.$$

Every point is an equilibrium point for this system.

12. True. If $\mathbf{A}$ is the matrix and λ is the eigenvalue associated to $\mathbf{Y}_0$, then

$$\mathbf{A}(k\mathbf{Y}_0) = k\mathbf{A}\mathbf{Y}_0 = k\lambda\mathbf{Y}_0 = \lambda(k\mathbf{Y}_0).$$

Consequently, $k\mathbf{Y}_0$ is an eigenvector as long as $k \neq 0$. (Note that $k = 0$ is excluded because the zero vector is never an eigenvector by definition.)

13. True. Linear systems have solutions that consist of just sine and cosine functions only when the eigenvalues are purely imaginary (that is, of the form $\pm i\omega$). In this case, the sine and cosine terms are of the form $\sin \omega t$ and $\cos \omega t$. For the first coordinate of $\mathbf{Y}(t)$ to be part of a solution, we would have to have $\omega = 2$, but the second coordinate would force $\omega = 1$. So this function cannot be the solution of a linear system.

15. False. In the graph, the amount of time between consecutive crossings of the t-axis decreases as t increases. Even though solutions of underdamped harmonic oscillators oscillate, the amount of time between consecutive crossings of the t-axis is constant.

19. First, we compute the characteristic polynomials and eigenvalues for each matrix.

(i) The characteristic polynomial is $\lambda^2 + 1$, and the eigenvalues are $\lambda = \pm i$. Center.

(ii) The characteristic polynomial is $\lambda^2 + 2\lambda - 2$, and the eigenvalues are $\lambda = 1 \pm \sqrt{3}$. Saddle.

(iii) The characteristic polynomial is $\lambda^2 + 3\lambda + 1$, and the eigenvalues are $\lambda = (-3 \pm \sqrt{5})/2$. Sink.

(iv) The characteristic polynomial is $\lambda^2 + 1$, and the eigenvalues are $\lambda = \pm i$. Center.

(v) The characteristic polynomial is $\lambda^2 - \lambda - 2$, and the eigenvalues are $\lambda = -1$ and $\lambda = 2$. Saddle.

(vi) The characteristic polynomial is $\lambda^2 - 3\lambda + 1$, and the eigenvalues are $\lambda = (3 \pm \sqrt{5})/2$. Source.

(vii) The characteristic polynomial is $\lambda^2 + 4\lambda + 4$. The eigenvalue $\lambda = -2$ is a repeated eigenvalue. Sink.

(viii) The characteristic polynomial is $\lambda^2 + 2\lambda + 3$, and the eigenvalues are $\lambda = -1 \pm i\sqrt{2}$. Spiral sink.

Given this information, we can match the matrices with the phase portraits.

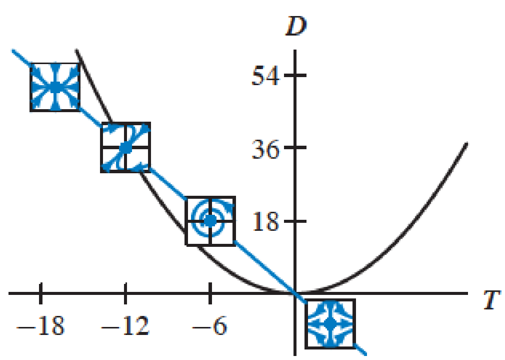
(a) This portrait is a center. There are two possibilities, (i) and (iv). At $(1, 0)$, the vector for (i) is $(1, -2)$, and the vector for (iv) is $(-1, -2)$. This phase portrait corresponds to matrix (iv).

(b) This portrait is a sink with two lines of eigenvectors. The only possibility is matrix (iii).

(c) This portrait is a saddle. The only possibilities are (ii) and (v). However, in (v), all vectors on the y-axis are eigenvectors corresponding to the eigenvalue $\lambda = -1$. Therefore, the phase portrait cannot correspond to (v).

(d) This portrait is a sink with a single line of eigenvectors. The only possibility is matrix (vii).

20. **(a)** The trace T is a, and the determinant D is $-3a$. Therefore, the curve in the trace-determinant plane is $D = -3T$.



(b) The line $D = -3T$ crosses the parabola $T^2 - 4D = 0$ at two points—at $(T, D) = (-12, 36)$ if $a = -12$ and at $(T, D) = (0, 0)$ if $a = 0$. Therefore, bifucations occur at $a = -12$ and at $a = 0$. The portion of the line for which $a < -12$ corresponds to a positive determinant and a negative trace such that $T^2 - 4D < 0$. The corresponding phase portraits are real sinks. If $a = -12$, we have a sink with repeated eigenvalues. If $-12 < a < 0$, we have complex eigenvalues with negative real parts. Therefore, the phase portraits are spiral sinks. If $a = 0$, we have a degenerate case where the y-axis is an entire line of equilibrium points. Finally, if $a > 0$, the corresponding portion of the line is below the T-axis, and the phase portraits are saddles.

21. First, we compute the characteristic polynomials and eigenvalues for each matrix.

(i) The characteristic polynomial is $\lambda^2 - 3\lambda - 4$, and the eigenvalues are $\lambda = -1$ and $\lambda = 4$. Saddle.

(ii) The characteristic polynomial is $\lambda^2 - 7\lambda + 10$, and the eigenvalues are $\lambda = 2$ and $\lambda = 5$. Source.

(iii) The characteristic polynomial is $\lambda^2 + 4\lambda + 3$, and the eigenvalues are $\lambda = -3$ and $\lambda = -1$. Sink.

(iv) The characteristic polynomial is $\lambda^2 + 4$, and the eigenvalues are $\lambda = \pm 2i$. Center.

(v) The characteristic polynomial is $\lambda^2 + 9$, and the eigenvalues are $\lambda = \pm 3i$. Center.

(vi) The characteristic polynomial is $\lambda^2 - 2\lambda + \frac{15}{16}$, and the eigenvalues are $\lambda = 3/4$ and $\lambda = 5/4$. Source.

(vii) The characteristic polynomial is $\lambda^2 + 2.2\lambda + 5.21$. The eigenvalues are $\lambda = -1.1 \pm 2i$. Spiral sink.

(viii) The characteristic polynomial is $\lambda^2 + 0.2\lambda + 4.01$, and the eigenvalues are $\lambda = -0.1 \pm 2i$. Spiral sink.

Given this information, we can match the matrices with the $x(t)$- and $y(t)$-graphs.

(a) This solution approaches equilibrium without oscillating. Therefore, the system has at least one negative real eigenvalue. Matrices (i) and (iii) are the only matrices with a negative eigenvalue. Since matrix (i) corresponds to a saddle, its only solutions that approach equilibrium are straight line-solutions. However, this solution is not a straight-line solution because $y(t)/x(t)$ is not constant. This solution must correspond to matrix (iii).

(b) Note that $y(t) = -x(t)$ for all t. Therefore, this solution corresponds to a straight-line solution for a source or a saddle with eigenvector $(1, -1)$. Direct computation shows that $(1, -1)$ is not an eigenvector for matrices (i) and (ii), and it is an eigenvector corresponding to eigenvalue $\lambda = 3/4$ for matrix (vi).

(c) This solution is periodic. Therefore, the corresponding matrix has purely imaginary eigenvalues. Matrices (iv) and (v) are the only matrices with purely imaginary eigenvalues. The solution oscillates three times over any interval of length 2π. Hence, the period of the solution is $2\pi/3$. Therefore, the eigenvalues must be $\pm 3i$, and this solution corresponds to matrix (v).

(d) This solution oscillates as it approaches equilibrium. Therefore, the corresponding matrix has complex eigenvalues with a negative real part. Matrices (vii) and (viii) are the only possibilities. Since the real part of the eigenvalues for matrix (vii) is -1.1, its solutions decay at a rate of $e^{-1.1t}$. Similarly, the real part of the eigenvalues for matrix (viii) is -0.1, so its solutions decay at a rate of $e^{-0.1t}$. The rate of decay of the solution graphed is $e^{-0.1t}$. Consequently, these graphs correspond to matrix (viii).

23. The characteristic polynomial is

$$s^2 + 5s + 6,$$

so the eigenvalues are $s = -2$ and $s = -3$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-3t},$$

and we have

$$y'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_1 - 3k_2 = 2. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 2$ and $k_2 = -2$. Hence, the solution to our initial-value problem is $y(t) = 2e^{-2t} - 2e^{-3t}$.

24. The characteristic polynomial is

$$s^2 + 2s + 5,$$

so the eigenvalues are $s = -1 \pm 2i$. Hence, the general solution is

$$y(t) = k_1 e^{-t} \cos 2t + k_2 e^{-t} \sin 2t.$$

From the initial condition $y(0) = 3$, we see that $k_1 = 3$. Differentiating

$$y(t) = 3e^{-t} \cos 2t + k_2 e^{-t} \sin 2t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -3 + 2k_2$. Since $y'(0) = -1$, we have $k_2 = 1$. Hence, the solution to our initial-value problem is

$$y(t) = 3e^{-t} \cos 2t + e^{-t} \sin 2t.$$

25. The characteristic polynomial is

$$s^2 + 2s + 1,$$

so $s = -1$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-t} + k_2 t e^{-t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating

$$y(t) = e^{-t} + k_2 t e^{-t}$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -1 + k_2$. Since $y'(0) = 1$, we have $k_2 = 2$. Hence, the solution to our initial-value problem is

$$y(t) = e^{-t} + 2te^{-t}.$$

26. The characteristic polynomial is s^2+2, so the eigenvalues are $s = \pm i\sqrt{2}$. Hence, the general solution is

$$y(t) = k_1 \cos \sqrt{2}\, t + k_2 \sin \sqrt{2}\, t.$$

From the initial condition $y(0) = 3$, we see that $k_1 = 3$. Differentiating $y(t)$ and evaluating at $t = 0$, we get $y'(0) = \sqrt{2}\, k_2$. Since $y'(0) = -\sqrt{2}$, we have $k_2 = -1$. The solution to our initial-value problem is

$$y(t) = 3 \cos \sqrt{2}\, t - \sin \sqrt{2}\, t.$$

27. **(a)** The characteristic polynomial is

$$(1 - \lambda)(-1 - \lambda) - 3 = \lambda^2 - 4$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$. The equilibrium point at the origin is a saddle.

The eigenvectors for $\lambda_1 = 2$ satisfy the equations

$$\begin{cases} x + 3y = 2x \\ x - y = 2y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $x = 3y$. The eigenvector $(3, 1)$ is one such point.

The eigenvectors for $\lambda_2 = -2$ satisfy the equations

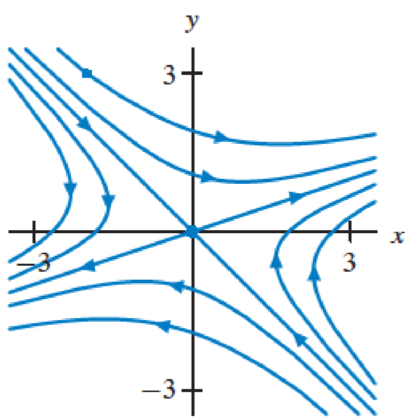
$$\begin{cases} x + 3y = -2x \\ x - y = -2y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $y = -x$. The eigenvector $(1, -1)$ is one such point.

Hence, the general solution of the system is

$$\mathbf{Y}(t) = k_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b)



(c) To solve the initial-value problem, we solve for k_1 and k_2 in the equation

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

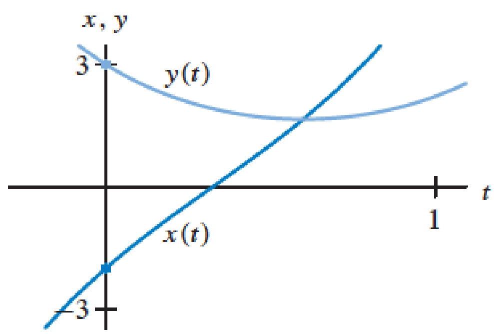
This vector equation is equivalent to the two scalar equations

$$\begin{cases} 3k_1 + k_2 = -2 \\ k_1 - k_2 = 3, \end{cases}$$

so $k_1 = 1/4$ and $k_2 = -11/4$. The solution to the initial-value problem is

$$\mathbf{Y}(t) = \tfrac{1}{4}e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \tfrac{11}{4}e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(d)



28. **(a)** The characteristic polynomial is

$$(4-\lambda)(3-\lambda)-2=\lambda^2-7\lambda+10=(\lambda-5)(\lambda-2),$$

so the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$. The equilibrium point at the origin is a source.

The eigenvectors for $\lambda_1 = 5$ satisfy the equations

$$\begin{cases} 4x + 2y = 5x \\ x + 3y = 5y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $x = 2y$. The eigenvector $(2, 1)$ is one such point.

The eigenvectors for $\lambda_2 = 2$ satisfy the equations

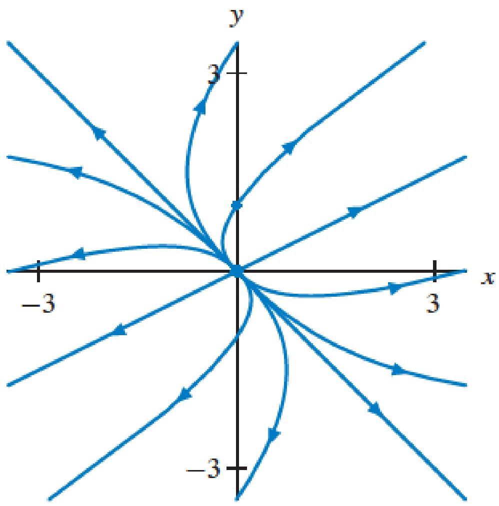
$$\begin{cases} 4x + 2y = 2x \\ x + 3y = 2y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $y = -x$. The eigenvector $(1, -1)$ is one such point.

Hence, the general solution is

$$\mathbf{Y}(t) = k_1 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b)



(c) To solve the initial-value problem, we solve for k_1 and k_2 in the equation

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

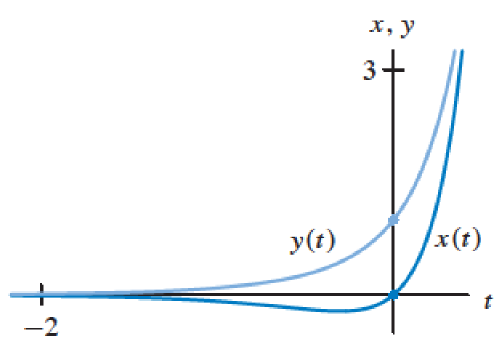
This vector equation is equivalent to the two scalar equations

$$\begin{cases} 2k_1 + k_2 = 0 \\ k_1 - k_2 = 1, \end{cases}$$

so $k_1 = 1/3$ and $k_2 = -2/3$. The solution of the initial-value problem is

$$\mathbf{Y}(t) = \tfrac{1}{3} e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \tfrac{2}{3} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(d)



30. **(a)** The characteristic polynomial is

$$(-3-\lambda)(1-\lambda)+12=\lambda^2+2\lambda+9,$$

so the eigenvalues are $\lambda=-1\pm 2\sqrt{2}\,i$. The equilibrium point at the origin is a spiral sink.

The eigenvectors (x, y) corresponding to $\lambda=-1+2\sqrt{2}\,i$ are solutions of the equations

$$\begin{cases} -3x+6y=(-1+2\sqrt{2}\,i)x \\ -2x+y=(-1+2\sqrt{2}\,i)y. \end{cases}$$

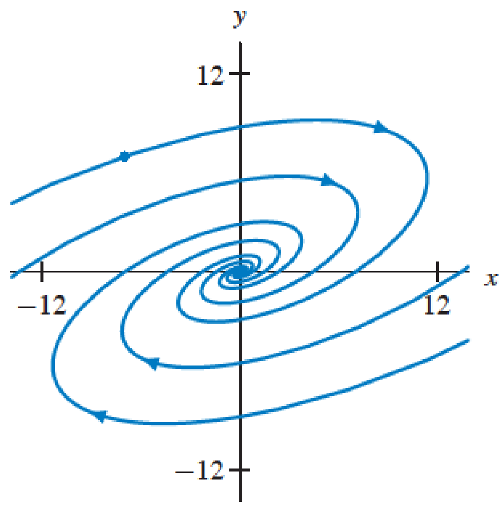
These equations are equivalent to the equation $3y=(1+\sqrt{2}\,i)x$. Consequently, $(3, 1+\sqrt{2}\,i)$ is one eigenvector. Linearly independent solutions are given by the real and imaginary parts of

$$\mathbf{Y}_c(t)=e^{(-1+2\sqrt{2}\,i)t}\begin{pmatrix} 3 \\ 1+\sqrt{2}\,i \end{pmatrix}.$$

Hence, the general solution is

$$\mathbf{Y}(t)=k_1e^{-t}\begin{pmatrix} 3\cos 2\sqrt{2}\,t \\ \cos 2\sqrt{2}\,t-\sqrt{2}\sin 2\sqrt{2}\,t \end{pmatrix}+k_2e^{-t}\begin{pmatrix} 3\sin 2\sqrt{2}\,t \\ \sqrt{2}\,\cos 2\sqrt{2}\,t+\sin 2\sqrt{2}\,t \end{pmatrix}.$$

(b)



(c) To satisfy the initial condition, we solve

$$\begin{pmatrix} -7 \\ 7 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix},$$

which is equivalent to the two scalar equations

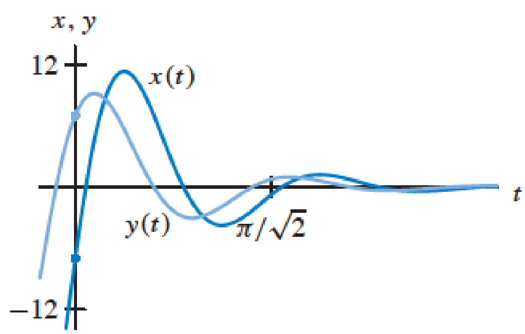
$$\begin{cases} 3k_1 = -7 \\ k_1 + \sqrt{2}\,k_2 = 7. \end{cases}$$

We get $k_1 = -7/3$ and $k_2 = 14\sqrt{2}/3$. The solution of the initial-value problem is

$$\mathbf{Y}(t) = -\frac{7}{3}e^{-t} \begin{pmatrix} 3\cos 2\sqrt{2}\,t \\ \cos 2\sqrt{2}\,t - \sqrt{2}\sin 2\sqrt{2}\,t \end{pmatrix} + \frac{14\sqrt{2}}{3}e^{-t} \begin{pmatrix} 3\sin 2\sqrt{2}\,t \\ \sqrt{2}\cos 2\sqrt{2}\,t + \sin 2\sqrt{2}\,t \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} -7\cos 2\sqrt{2}\,t + 14\sqrt{2}\,\sin 2\sqrt{2}\,t \\ 7\cos 2\sqrt{2}\,t + 7\sqrt{2}\,\sin 2\sqrt{2}\,t \end{pmatrix}.$$

(d)



31. **(a)** The characteristic polynomial is

$$(-3-\lambda)(-1-\lambda)+1=(\lambda+2)^2,$$

so $\lambda=-2$ is a repeated eigenvalue. The equilibrium point at the origin is a sink.

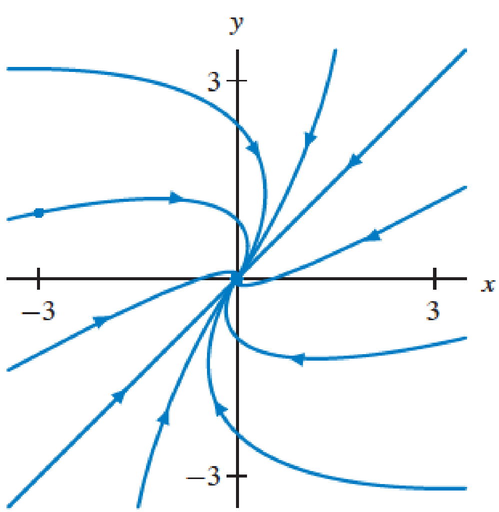
To find the general solution, we start with an arbitrary initial condition (x_0, y_0), and we calculate

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}=\begin{pmatrix} y_0-x_0 \\ y_0-x_0 \end{pmatrix}.$$

We obtain the general solution

$$\mathbf{Y}(t)=e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}+te^{-2t}\begin{pmatrix} y_0-x_0 \\ y_0-x_0 \end{pmatrix}.$$

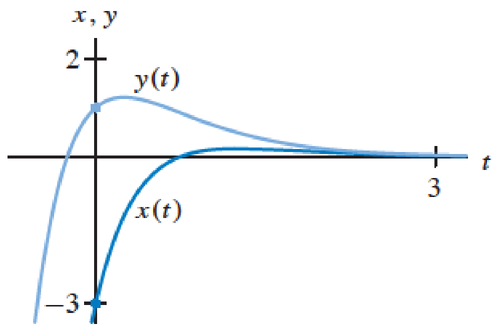
(b)



(c) The solution that satisfies the initial condition $(x_0, y_0) = (-3, 1)$ is

$$\mathbf{Y}(t) = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + te^{-2t}\begin{pmatrix} 4 \\ 4 \end{pmatrix} = e^{-2t}\begin{pmatrix} 4t-3 \\ 4t+1 \end{pmatrix}.$$

(d)



32. **(a)** The characteristic polynomial is

$$(0-\lambda)(-3-\lambda)+2=\lambda^2+3\lambda+2,$$

so the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-1$. The equilibrium point at the origin is a sink.

The eigenvectors for $\lambda_1=-2$ satisfy the equations

$$\begin{cases} y=-2x \\ -2x-3y=-2y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $y=-2x$. The eigenvector $(1,-2)$ is one such point.

The eigenvectors for $\lambda_2=-1$ satisfy the equations

$$\begin{cases} y=-x \\ -2x-3y=-y. \end{cases}$$

Consequently, the eigenvectors (x, y) for this eigenvalue satisfy $y=-x$. The eigenvector $(1,-1)$ is one such point.

Hence, the general solution of the system is

$$\mathbf{Y}(t)=k_1e^{-2t}\begin{pmatrix}1\\-2\end{pmatrix}+k_2e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

(b)