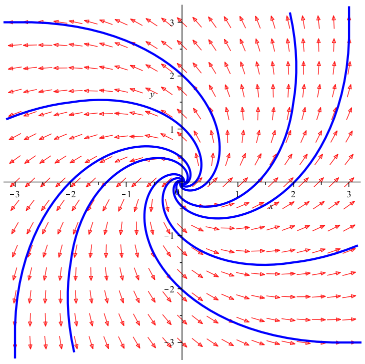
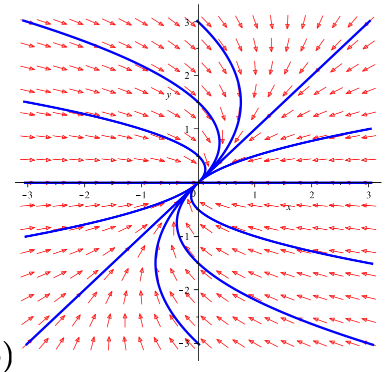
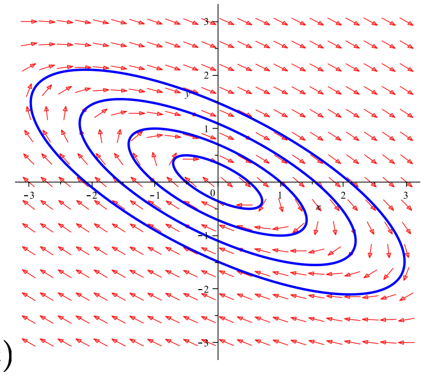
MATH 304 Ordinary Differential Equations

Exam #2 SOLUTIONS November 20, 2014 Prof. G. Roberts

1. Below are three phase portraits for three different linear systems $\dot{\mathbf{Y}} = A\mathbf{Y}$. Match each diagram with its corresponding matrix A. Explain briefly the rationale for each choice. There is exactly one matrix for each figure. (18 pts.)

(a)  **(b)**  **(c)** 

(i) $\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(a) <u>**(vi)**</u> This phase portrait corresponds to a spiral source. The matrices for **(v)** and **(vi)** each have trace $T = 2$ and determinant $D = 2$. Thus, $D > T^2/4$ and $T > 0$, so each matrix is a spiral source. Checking the direction field, we notice that the matrix for **(v)** applied to the vector $[1, 0]$ gives $[1, -1]$, which does not agree with the direction field at the point $(1, 0)$. On the other hand, **(vi)** applied to the vector $[1, 0]$ gives $[1, 1]$, which does match the direction field at $(1, 0)$.

(b) <u>**(i)**</u> This phase portrait corresponds to a real sink with two straight-line solutions, one along the x-axis (vector $[1, 0]$) and one along the line $y = x$ (vector $[1, 1]$). The matrices for **(i)** and **(ii)** each have eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$ (they are triangular so the eigenvalues are sitting on the diagonal), and thus correspond to real sinks. However, the matrix for **(ii)** applied to the vector $[1, 1]$ gives $[0, -2]$, which means it is not an eigenvector. Meanwhile, **(i)** applied to the vector $[1, 1]$ gives $[-1, -1]$, which means that $[1, 1]$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ (the slower eigenvalue). It follows that solutions for **(i)** will come in tangent to the line $y = x$, as desired.

(c) <u>**(iv)**</u> This phase portrait corresponds to a center. The matrices for **(iii)** and **(iv)** each have trace $T = 0$ and determinant $D = 1$, which corresponds to a center. Checking the direction field, we notice that the matrix for **(iii)** applied to the vector $[1, 0]$ gives $[-1, -1]$, which does not agree with the direction field at the point $(1, 0)$. On the other hand, **(iv)** applied to the vector $[1, 0]$ gives $[1, -1]$, which does match the direction field at $(1, 0)$.

2. **ODE Potpourrii:** (16 pts.)

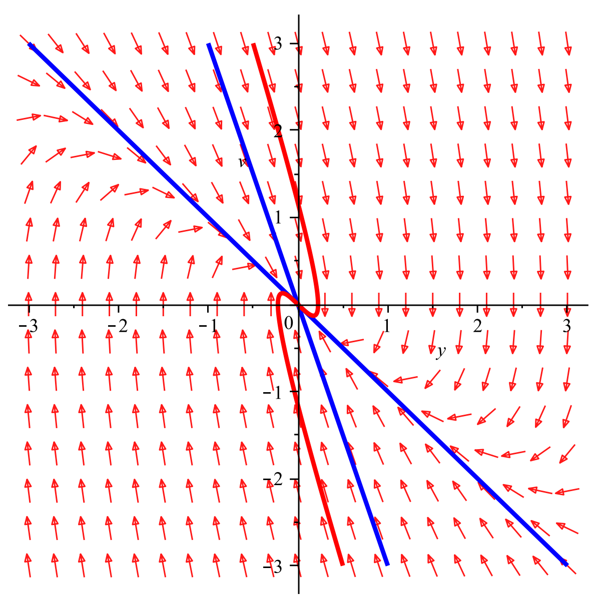
(a) Suppose that a harmonic oscillator has a mass $m = 2$ and a spring constant $k = 5$. For what value of the damping coefficient b is the oscillator critically damped?

Answer: $b = \sqrt{40} = 2\sqrt{10}$. The ODE for the oscillator is $2\ddot{y} + b\dot{y} + 5y = 0$, which means the characteristic polynomial is $p(\lambda) = 2\lambda^2 + b\lambda + 5$. For the oscillator to be critically damped, we want p to have repeated roots. This means that the discriminant of the quadratic must vanish, i.e., $b^2 - 4 \cdot 2 \cdot 5 = 0$, or $b^2 = 40$. We take the positive square root because the damping coefficient is assumed to be positive.

(b) True or False: An overdamped harmonic oscillator always returns to the equilibrium position without crossing the rest position. Explain briefly.

Answer: False. This was a tricky one. Note that the rest position is $y = 0$, which is different than the equilibrium *point* $y = 0, v = 0$. An overdamped oscillator always heads toward the equilibrium point without repeated oscillation. However, if the initial velocity is large enough and in the opposite direction of the initial position, then the mass can move past the rest position (y flips signs) before heading toward equilibrium. We discussed this on an exercise Melissa asked about during the exam review session.

Here's a more detailed explanation. For an overdamped oscillator, there are two real, negative eigenvalues $\lambda_2 < \lambda_1 < 0$. Since the top row of the matrix of the corresponding linear system is $[0\ 1]$, the eigenvectors can always be chosen as $v_1 = [1\ \lambda_1]$ and $v_2 = [1\ \lambda_2]$. Next, recall that solutions for a real sink come in tangent to the *slower* eigendirection, which in our setup is the vector v_1. In the fourth quadrant of the yv-plane, the vector v_1 will always be above the vector v_2. Thus, any solution that starts in the fourth quadrant below the v_2 straight-line solution, must cross into the third and second quadrants before heading toward the origin. This means that $y(0) > 0$, but that $y(t) < 0$ as the solution approaches the equilibrium point. Consequently, the mass passes through the rest position $y = 0$ (just once) before returning to equilibrium. The figure below shows an overdamped oscillator with $\lambda_1 = -1, \lambda_2 = -3$ and two solution curves (in red) where $y(t)$ changes sign as it heads toward $(0, 0)$.



(c) Find the natural period of the oscillations for solutions to the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 1 \\ -5 & 4 \end{bmatrix} \mathbf{Y}.$$

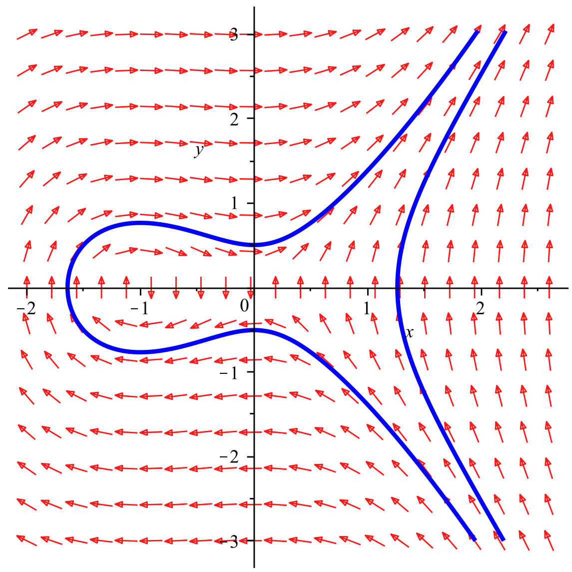
Answer: π. To find the natural period, we first compute the eigenvalues. The trace is $T = 6$ and the determinant is $D = 13$, so the characteristic polynomial is $p(\lambda) = \lambda^2 - 6\lambda + 13$. The roots are $(6 \pm \sqrt{36 - 52})/2 = 3 \pm 2\,i$. The coefficient of the imaginary part controls the natural period of the solution, so the period is simply $2\pi/2 = \pi$.

(d) Find **one** solution to the system

$$\begin{aligned} \frac{dx}{dt} &= (y+3)(e^{x^2+5} + \tan^2 y) \\ \frac{dy}{dt} &= x(y^{2014} + 4). \end{aligned}$$

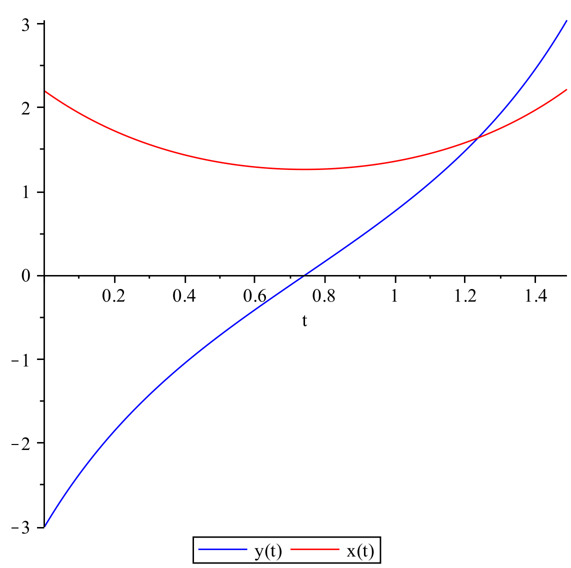
Answer: $x(t) = 0, y(t) = -3$ is an equilibrium point, and thus a simple solution to the ODE.

3. The phase portrait below shows two solution curves to a given system of differential equations. Solution curve **(a)** begins at the point $(x(0), y(0)) = (2.2, -3)$, and solution curve **(b)** starts at the point $(x(0), y(0)) = (1.95, -3)$. (14 pts.)



(a) Sketch the $x(t)$- and $y(t)$-graphs for solution curve **(a)** on the **same** set of axes. Be sure to label each graph.

(b) Sketch the $x(t)$- and $y(t)$-graphs for solution curve **(b)** on the **same** set of axes. Be sure to label each graph.



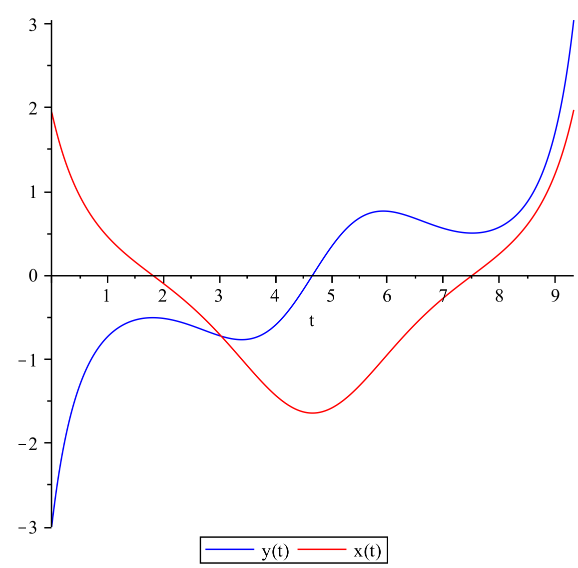


Figure 1: The $x(t)$- and $y(t)$-graphs for solution curve **(a)** (left) and **(b)** (right). Note that in each figure, the minimum value of x occurs when $y = 0$.

4. Consider the linear system $\dfrac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ with

$$A = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Sketch the phase portrait for this system, including several solution curves.

(c) Find the general solution of the system.

(d) On the **same** set of axes, sketch the $x(t)$- and $y(t)$-graphs for the solution with initial condition $\mathbf{Y}_0 = (-2, 4)$. Be sure to label each graph.

(18 pts.)

Answer: The trace of the matrix is $T = 3$ and the determinant is $D = -10$. This gives a characteristic polynomial of

$$p(\lambda) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2),$$

so the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 5$ and the equilibrium point is a saddle. To find the eigenvectors, we have

$$A - \lambda_1 I = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \quad \Longrightarrow \quad v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

and

$$A - \lambda_2 I = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \quad \Longrightarrow \quad v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

It follows that the general solution is

$$\mathbf{Y}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants.

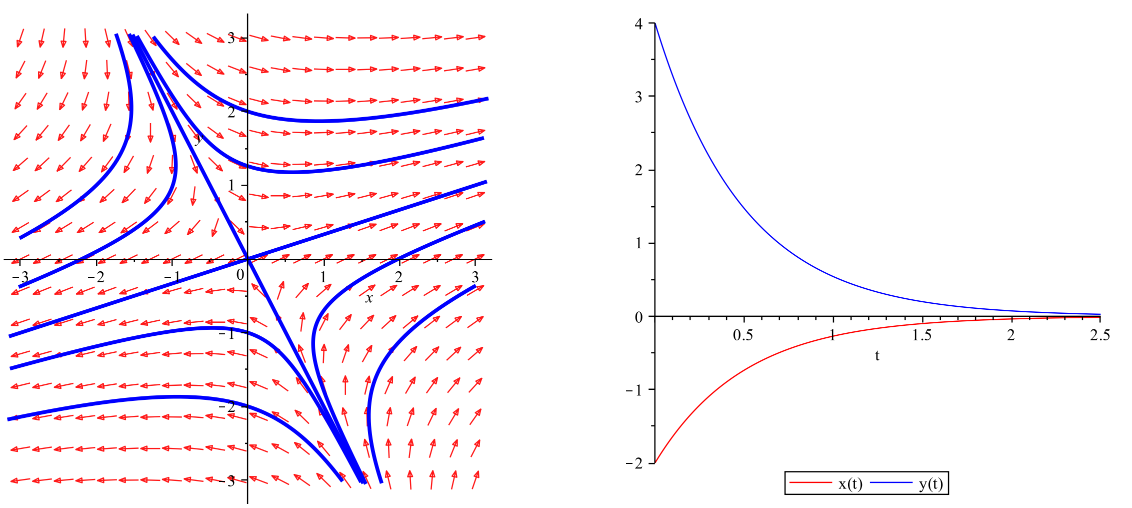


Figure 2: The phase portrait for the given linear system (left) and the $x(t)$- and $y(t)$-graphs (right) for the solution starting at the point $(-2, 4) = -2v_1$ (a straight-line solution heading to the origin).

5. Consider the following harmonic oscillator:

$$\ddot{y} + 2\dot{y} + 10y \;=\; 0.$$

(a) Find the general solution, that is, find a formula for $y(t)$, of this second-order differential equation. (5 pts.)

Answer: The characteristic polynomial is $p(\lambda) = \lambda^2 + 2\lambda + 10$, which has roots

$$\lambda \;=\; \frac{-2 \pm \sqrt{4-40}}{2} \;=\; -1 \pm 3\,i.$$

Since the real part of the eigenvalues is $a = -1$ and the imaginary part is $b = 3$, the general solution is

$$y(t) \;=\; c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t).$$

(b) Find the particular solution that satisfies the initial conditions $y(0) = 4, v(0) = 0$. (5 pts.)

Answer: Using our general solution from part **(a)**, we take the derivative and find that

$$y'(t) \;=\; v(t) \;=\; c_1(-e^{-t}\cos(3t) - 3e^{-t}\sin(3t)) + c_2(-e^{-t}\sin(3t) + 3e^{-t}\cos(3t)).$$

Then, $y(0) = 4$ implies that $4 = c_1 + 0$, so $c_1 = 4$. Next, $v(0) = 0$ implies that $0 = -c_1 + 3c_2$, or $4 = 3c_2$, so $c_2 = 4/3$. Thus, the particular solution is

$$y(t) \;=\; 4e^{-t}\cos(3t) + \frac{4}{3}e^{-t}\sin(3t).$$

(c) Carefully describe the motion of the mass for the initial conditions $y(0) = 4, v(0) = 0$. Be sure to include an interpretation of the initial conditions and a description of how the mass behaves as it approaches the rest position. (How often, if at all, does the mass cross the rest position? How fast does it approach equilibrium?) (4 pts.)

Answer: The mass begins four units from the rest position with no initial velocity (no movement). It will oscillate about the rest position repeatedly but with smaller and smaller amplitudes. The maximum and minimum distances from $y = 0$ are governed by the envelope e^{-t}. Since the natural period is $2\pi/3$, the mass crosses the rest position twice every $2\pi/3$ seconds. It approaches the equilibrium exponentially at a rate of e^{-t}.

6. Consider the one-parameter family of linear systems given by

$$\frac{d\mathbf{Y}}{dt} \;=\; \begin{bmatrix} 2a & a-1 \\ a+1 & 0 \end{bmatrix}\mathbf{Y}.$$

(a) Give the trace T and determinant D of this matrix, and find a relationship between T and D.

Answer: The trace is $T = 2a$ and the determinant is $D = -(a+1)(a-1) = 1 - a^2$. Solving the first equation for a gives $a = T/2$. Then, substituting this into the second equation, we have $D = 1 - (T/2)^2 = 1 - T^2/4$.

(b) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

Answer: The function $D = 1 - T^2/4$ is a parabola opening downwards with a vertex at $(T = 0, D = 1)$ (see Figure on next page).

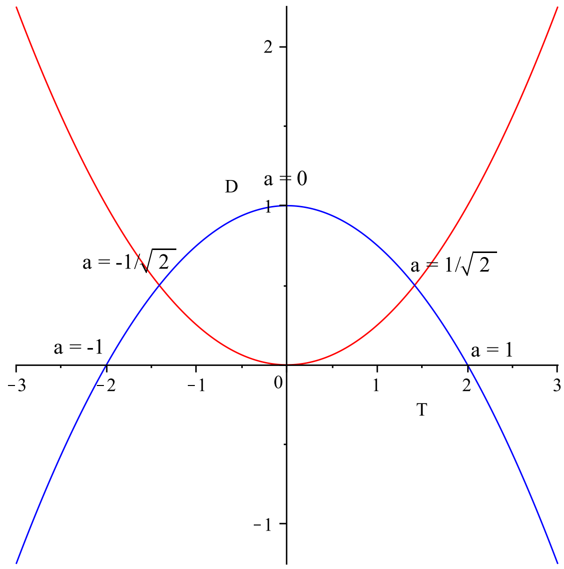


Figure 3: The graph of the curve $D = 1 - T^2/4$ (blue curve) in the trace-determinant plane. The red curve is the repeated eigenvalue parabola $D = T^2/4$.

(c) Find all possible values of a where bifurcations occur. Describe the type of equilibrium point before, at, and after each bifurcation.

(20 pts.)

Answer: Bifurcations occur when the curve crosses the line $D = 0$, the parabola $D = T^2/4$, and the top half of the D-axis, $T = 0, D > 0$. First, we have that $D = 0$ iff $1 - a^2 = 0$ or $a = \pm 1$. Next, the parabolas $D = T^2/4$ and $D = 1 - T^2/4$ intersect when $T^2/4 = 1 - T^2/4$, or $T^2 = 2$. Since $T = 2a$, we get $4a^2 = 2$ or $a = \pm 1/\sqrt{2}$. Finally, we have $T = 0$ when $a = 0$. Thus, there are five bifurcations at $a = -1, -1/\sqrt{2}, 0, 1/\sqrt{2}, 1$ (see Figure). Moving around the parabola from $a < -1$ to $a > 1$, we have the following types of equilibria at the origin:

- For $a < -1$, we have a saddle.
- At $a = -1$, there is a line of equilibrium points on $y = -x$. All other solutions move toward this line parallel to the x-axis.
- For $-1 < a < -1/\sqrt{2}$, we have a real sink.
- At $a = -1/\sqrt{2}$, we have a repeated sink.
- For $-1/\sqrt{2} < a < 0$, we have a spiral sink.
- At $a = 0$, we have a center.
- For $0 < a < 1/\sqrt{2}$, we have a spiral source.
- At $a = 1/\sqrt{2}$, we have a repeated source.
- For $1/\sqrt{2} < a < 1$, we have a real source.
- At $a = 1$, there is a line of equilibrium points on the y-axis. All other solutions move away from this line parallel to the line $y = x$.
- For $a > 1$, we have a saddle.