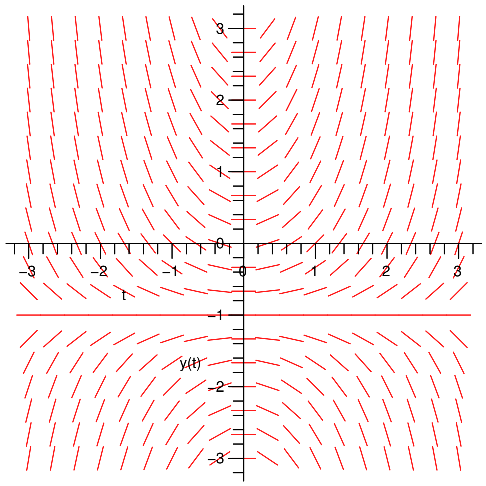
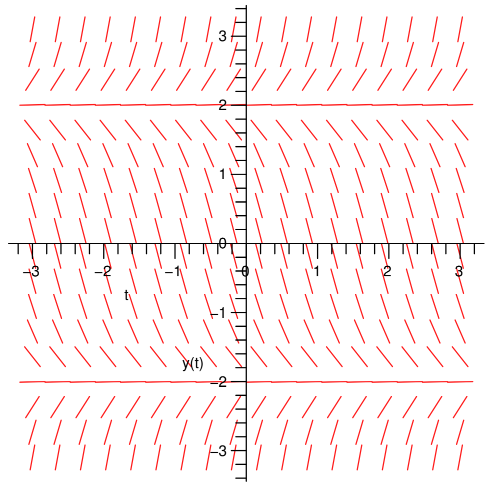
MATH 304 Ordinary Differential Equations

Exam #1 Solutions **October 8, 2014** **Prof. G. Roberts**

1. Match each slope-field with the correct differential equation. There is exactly **one** ODE for each slope-field. Explain your choices with a brief sentence or two. (12 pts.)



(a)



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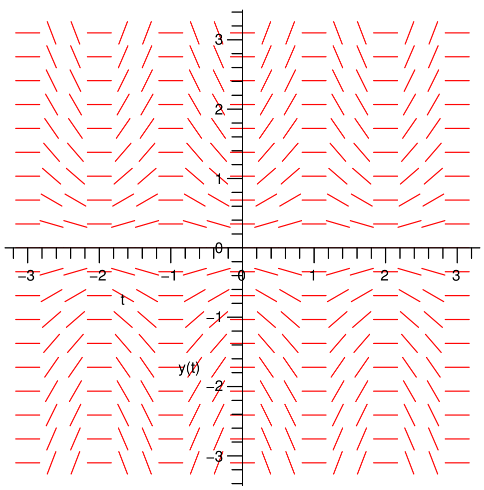
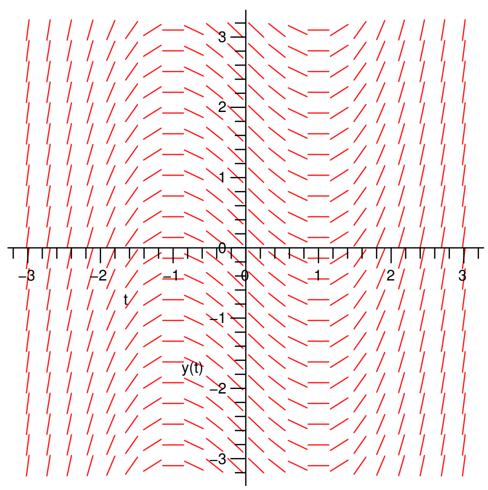


(c)



(d)

(**i**) $\dfrac{dy}{dt} = 4 - y^2$ (**iii**) $\dfrac{dy}{dt} = 1 - t^2$ (**v**) $\dfrac{dy}{dt} = y\sin(\pi t)$ (**vii**) $\dfrac{dy}{dt} = ty - t$

(**ii**) $\dfrac{dy}{dt} = y^2 - 4$ (**iv**) $\dfrac{dy}{dt} = t^2 - 1$ (**vi**) $\dfrac{dy}{dt} = y\cos(\pi t)$ (**viii**) $\dfrac{dy}{dt} = ty + t$

(a) **(viii)** This is the only ODE having $y = -1$ as an equilibrium solution.

(b) **(ii)** The slope-field is constant on horizontal lines so it corresponds to an autonomous ODE. It has equilibria at $y = 2$ and $y = -2$, and has a negative slope at $y = 0$. This matches equation **(ii)**.

(c) **(v)** The slopes are 0 when $t = 0, \pm 1, \pm 2, \pm 3$, as is the function $\sin(\pi t)$.

(d) **(iv)** Since slopes are constant on vertical lines, the ODE is of the form $dy/dt = f(t)$. Moreover, the slope is negative when $t = 0$, so this corresponds to equation **(iv)**.

2. ODE Potpourrii:

(a) Write down the ODE for the logistic population model with growth rate 0.3 and carrying capacity 20. (4 pts.)

Answer:

$$\frac{dp}{dt} = 0.3P\left(1 - \frac{P}{20}\right)$$

(b) Suppose that $\dfrac{dy}{dt} = f(t, y)$ is a periodic differential equation with period $T = 4$. If the general solution to the ODE is

$$y(t) = ke^{t/2} - \frac{2}{3}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{5}\cos\left(\frac{\pi}{2}t\right),$$

find a specific formula for the Poincaré map $P(y_0)$. (6 pts.)

Answer: First, letting $y(0) = y_0$, we have $y_0 = k + 1/5$ so that $k = y_0 - 1/5$. The general solution can thus be written as

$$y(t) = (y_0 - 1/5)e^{t/2} - \frac{2}{3}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{5}\cos\left(\frac{\pi}{2}t\right),$$

which means the Poincaré map $P(y_0)$, found by evaluating this solution at time $T = 4$, is

$$P(y_0) = y(4) = (y_0 - 1/5)e^2 + \frac{1}{5} = e^2 y_0 + \frac{1}{5}(1 - e^2).$$

(c) Given the autonomous differential equation $\dfrac{dy}{dt} = f(y)$, where f is a differentiable function with $f(3) = 0, f'(3) = 0, f''(3) = 0$ and $f'''(3) = -4$. Which of the following statements is correct? (6 pts.)

(a) $y = 3$ is a source.

(b) $y = 3$ is a sink.

(c) $y = 3$ is a node.

(d) $y = 3$ is not an equilibrium solution.

(e) $y = 3$ is an equilibrium solution but there is not enough information given to determine what type of equilibrium point it is.

Answer: **(b)** $y = 3$ is a sink. Since $f(3) = 0$, we know that $y = 3$ is an equilibrium point. Then, $f'(3) = 0$ means the slope of f is 0 at $y = 3$, while $f'''(3) < 0$ means that the second derivative $f''(y)$ is decreasing at $y = 3$. Since $f''(3) = 0$, it follows that $f''(y) > 0$ (concave up) for y slightly less than 3, while $f''(y) < 0$ (concave down) for y slightly greater than 3. This, in turn, means that $f(y) > 0$ for y slightly less than 3, and $f(y) < 0$ for y slightly greater than 3. Therefore, using a phase line, it follows that $y = 3$ is a sink. A sample function satisfying the conditions given in the problem is $f(y) = -\frac{2}{3}(y - 3)^3$ (see Figure 1).

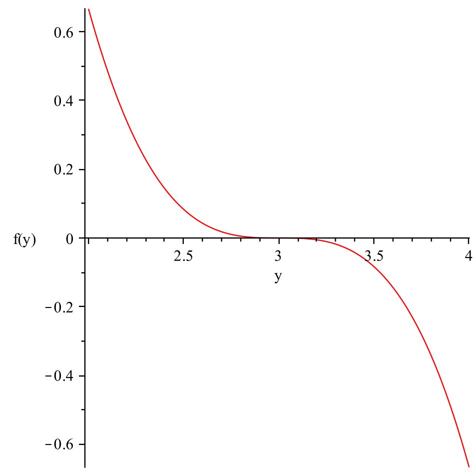


Figure 1: The graph of $f(y) = -\frac{2}{3}(y-3)^3$ near $y = 3$.

3. Consider the initial-value problem

$$\frac{dy}{dt} = 4ty^2, \quad y(0) = -1.$$

(a) Use Euler's method with $\Delta t = 0.5$ to approximate the solution (to three decimal places) over the time interval $0 \le t \le 2$. (8 pts.)

Answer: $y(2) \approx 0$. We repeatedly apply the formulas

$$\begin{aligned} t_{n+1} &= t_n + 0.5, \\ y_{n+1} &= y_n + m_n \cdot 0.5\ . \end{aligned}$$

The slope m_n is found at each stage by plugging the current t- and y-values into the right-hand side of the ODE: $f(t, y) = 4ty^2$. We obtain

n	t_n	y_n	m_n
0	0	-1	0
1	0.5	-1	2
2	1.0	0	0
3	1.5	0	0
4	2	0	

(b) Sketch a graph of your Euler's method solution and explain why it is clearly NOT a possible solution to the ODE. (4 pts.)

Answer: Note that both $f(t, y) = 4ty^2$ and $\partial f/\partial y = 8ty$ are continuous on the ty-plane, so the hypotheses of the Uniqueness Theorem hold. The graph in Figure 2 cannot be a solution to the ODE because it intersects the equilibrium solution $y(t) = 0$ at the point $(1, 0)$. By the Uniqueness Theorem, two solutions that agree at a point must be identical on some interval about that point. But our solution moves away from $y = 0$ if $t < 1$, contradicting the Uniqueness Theorem.

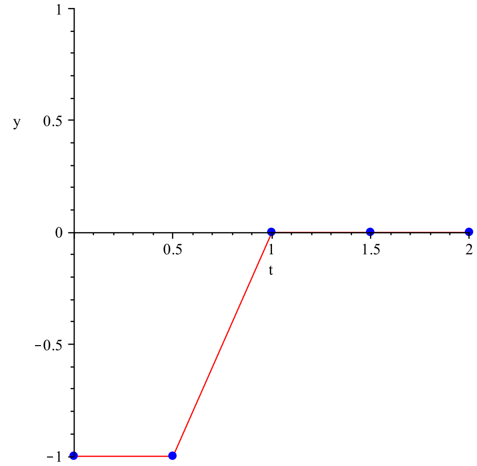


Figure 2: The graph of the Euler's method approximation.

4. Solve the given initial-value problem. (18 pts.)

(a) $\dfrac{dy}{dt} = \dfrac{4}{t}y + t^4\sin(\pi t), \quad y(1) = 3.$

Answer: $y(t) = -\dfrac{t^4}{\pi}\cos(\pi t) + (3 - 1/\pi)t^4$

We use the method of integrating factors. We have that

$$\mu(t) \;=\; e^{\int -\frac{4}{t}\,dt} \;=\; e^{-4\ln|t|} \;=\; e^{\ln|t|^{-4}} \;=\; t^{-4}.$$

Multiplying the ODE by t^{-4} gives

$$t^{-4}\cdot\frac{dy}{dt} - 4t^{-5}\cdot y \;=\; \sin(\pi t),$$

which is equivalent to

$$\left(y\cdot t^{-4}\right)' \;=\; \sin(\pi t).$$

Integrating both side of this equation gives

$$y\cdot t^{-4} \;=\; -\frac{1}{\pi}\cos(\pi t) + c,$$

so that $y = -\dfrac{t^4}{\pi}\cos(\pi t) + ct^4$. Since $y(1) = 3$, we have $3 = -\frac{1}{\pi}(-1) + c$ or $c = 3 - 1/\pi$. The final solution is thus

$$y(t) = -\frac{t^4}{\pi}\cos(\pi t) + (3 - 1/\pi)t^4.$$

(b) $\dfrac{dz}{dt} = 2z + 5\cos(4t), \quad z(0) = -3.$

Answer: $z(t) = -\frac{5}{2}e^{2t} - \frac{1}{2}\cos(4t) + \sin(4t)$

This is a linear, non-homogeneous, first-order equation. The solution to the homogeneous part is simply $z_h(t) = ce^{2t}$. To find a particular solution to the ODE, we use the method of undetermined coefficients and guess $z_p(t) = \alpha\cos(4t) + \beta\sin(4t)$. Substituting our guess into the ODE yields

$$-4\alpha\sin(4t) + 4\beta\cos(4t) \;=\; 2\alpha\cos(4t) + 2\beta\sin(4t) + 5\cos(4t).$$

Lining up the cosine and sine terms, we obtain the system of linear equations:

$$\begin{aligned} -4\alpha &= 2\beta \\ 4\beta &= 2\alpha + 5. \end{aligned}$$

Using standard algebra, the solution to this system is $\alpha = -1/2$ and $\beta = 1$. Then, by the Extended Linearity Principle, the general solution to the ODE is

$$z(t) = ce^{2t} - \frac{1}{2}\cos(4t) + \sin(4t).$$

The initial condition $z(0) = -3$ implies that $-3 = c - 1/2$ or $c = -5/2$. Therefore, the solution to the initial-value problem is

$$z(t) = -\frac{5}{2}e^{2t} - \frac{1}{2}\cos(4t) + \sin(4t).$$

5. Consider the initial-value problem

$$\frac{dy}{dt} \;=\; \sqrt{|1-y|}, \quad y(0) = 1.$$

(a) Find an "easy" solution to the initial-value problem. (3 pts.)

Answer: An easy solution satisfying both the ODE and the initial condition is the equilibrium solution $y(t) = 1$.

(b) Find another solution to the initial-value problem that intersects your solution from part **(a)**. *Hint:* This solution will be a piecewise function. (8 pts.)

Answer: The key here is to break the problem into two cases: $y > 1$ and $y < 1$, in order to handle the absolute value function. First, suppose that $y > 1$, so that $|1-y| = -(1-y) = y-1$. For this case, we must solve the ODE $dy/dt = \sqrt{y-1}$. Using the separate-and-integrate technique, we obtain

$$\frac{dy}{\sqrt{y-1}} \;=\; dt \quad\Longrightarrow\quad 2(y-1)^{1/2} \;=\; t + c \quad\Longrightarrow\quad \sqrt{y-1} \;=\; \frac{t}{2} + c.$$

Since we want a solution passing through the point $(0, 1)$, we substitute $t = 0$, $y = 1$ into the last equation to find that $c = 0$. Therefore, $\sqrt{y-1} = t/2$. Note that since $\sqrt{y-1} \geq 0$ by definition, we must also have that $t/2 \geq 0$ or $t \geq 0$. Solving for y yields

$$y(t) \;=\; \frac{t^2}{4} + 1 \quad \text{ if } t \geq 0.$$

Next, suppose that $y < 1$, so that $|1-y| = 1-y$. For this case, we must solve the ODE $dy/dt = \sqrt{1-y}$. Using the separate-and-integrate technique, we obtain

$$\frac{dy}{\sqrt{1-y}} = dt \quad \Longrightarrow \quad -2(1-y)^{1/2} = t + c \quad \Longrightarrow \quad \sqrt{1-y} = -\frac{t}{2} + c.$$

Since we want a solution passing through the point $(0, 1)$, we substitute $t = 0$, $y = 1$ into the last equation to find that $c = 0$. Therefore, $\sqrt{1-y} = -t/2$. Note that since $\sqrt{1-y} \geq 0$ by definition, we must also have that $-t/2 \geq 0$ or $t \leq 0$. Solving for y then yields

$$y(t) = -\frac{t^2}{4} + 1 \quad \text{if } t \leq 0.$$

Putting our two pieces together, we obtain the solution

$$y(t) = \begin{cases} \dfrac{t^2}{4} + 1 & \text{if } t \geq 0 \\ -\dfrac{t^2}{4} + 1 & \text{if } t < 0. \end{cases}$$

Note: It is worth checking carefully that this piecewise function satisfies the ODE. This is not the only possible solution. Choosing more general integration constants will yield an infinite family of solutions, as long as $y(0) = 1$ is specified.

(c) Why doesn't the existence of two solutions, each satisfying $y(0) = 1$, violate the Uniqueness Theorem? (4 pts.)

The graph of $f(y)$ has a cusp at $y = 1$ (see Figure 3) and is therefore not differentiable there. Thus, since $\partial f/\partial y$ is not continuous at $y = 1$ (it doesn't exist), the Uniqueness Theorem does not apply at the point $(t = 0, y = 1)$.

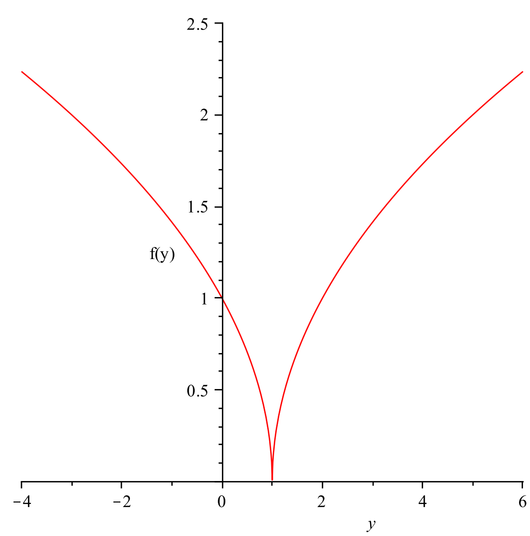


Figure 3: The graph of $f(y) = \sqrt{|1-y|}$.

6. Sketch the bifurcation diagram for the family of differential equations

$$\frac{dy}{dt} = y^4 + ay^2,$$

where a is a real parameter. Your graph should have a on the horizontal axis and y on the vertical axis, and should show any curves of equilibria. Identify all bifurcation values and describe the change in qualitative behavior of solutions before, at and after each bifurcation. (15 pts.)

Answer: We first solve to find the equilibrium points. We have $y^4+ay^2 = 0$ or $y^2(y^2+a) = 0$. Therefore, $y = 0$ is an equilibrium point for any value of a, and $y = \pm\sqrt{-a}$ are two equilibrium points provided that $a < 0$. It follows that $a = 0$ is a bifurcation value.

The bifurcation diagram is shown in Figure 4. Note that $f_a(y) = y^4 + ay^2$ is an even function in y, so the arrows will point in the same direction for $-y$ as they do for $+y$. The bifurcation at $a = 0$ is a pitchfork bifurcation.

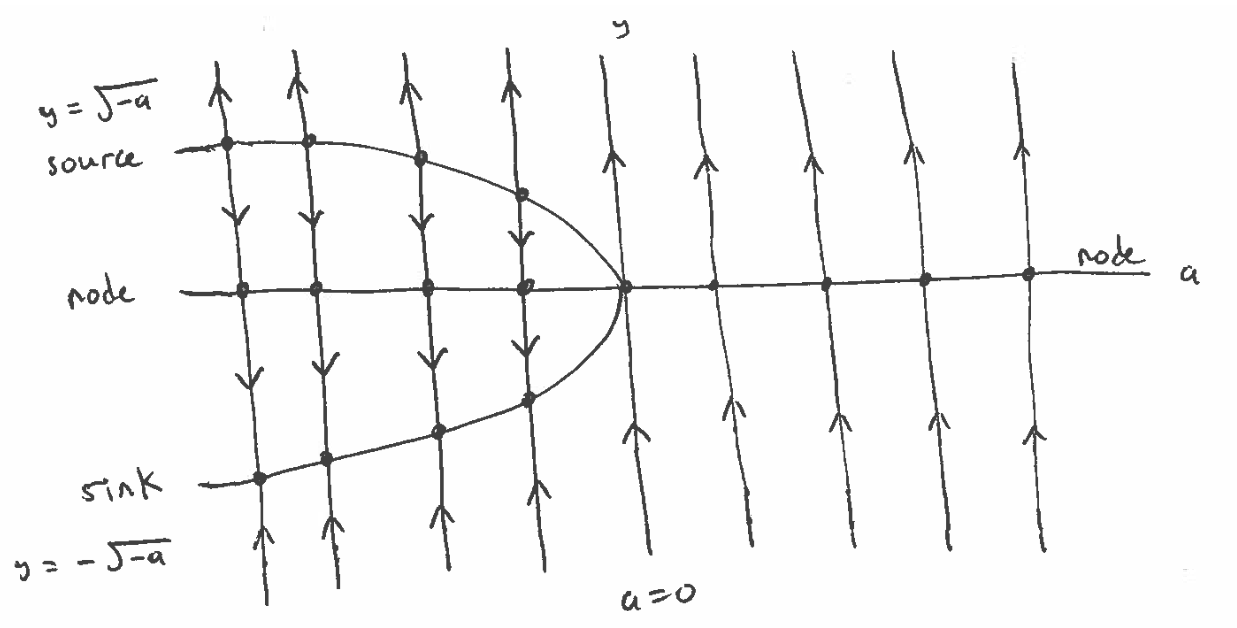


Figure 4: The bifurcation diagram for $dy/dt = y^4 + ay^2$ shows that a pitchfork bifurcation occurs at $a = 0$.

If $a > 0$, there is only one equilibrium at $y = 0$ and it is a node. Solutions are always increasing in this case, except for the equilibrium point. At $a = 0$, $y = 0$ is the only equilibrium and it is a node. For $a < 0$, $y = 0$ is still a node, but now solutions are decreasing to either side of it. Two new equilibrium points exist for this case, with $y = \sqrt{-a}$ a source, and $y = -\sqrt{-a}$ a sink.

7. Tanya and her family have saved \$120,000 for her college tuition. When she begins college, she invests the money in a high interest education-only savings account that pays 8% interest per year, compounded continuously. Her college tuition is \$40,000 each year, and Tanya has arranged to have her tuition money deducted in small amounts on a frequent basis. For this problem, assume that these deductions occur continuously. (12 pts.)

   (a) Let $y(t)$ represent the amount of money in Tanya's account after t years. Write down a differential equation for $y(t)$ that models the amount of money in her account.

   Answer:

$$\frac{dy}{dt} \;=\; 0.08y - 40,000, \qquad y(0) = 120,000.$$

   (b) Using your model, determine if Tanya will have enough money to finish college in four years. If she makes it, how much money does she have left in the account after graduation? If she doesn't make it, when does her money run out? Round your answer to two decimal places.

Answer: This is a linear, first-order, non-homogeneous ODE. The homogeneous solution is $y_h(t) = ce^{0.08t}$. For a particular solution, we guess $y_p(t) = k$, which leads to $0 = 0.08k - 40,000$ or $k = 500,000$. This is the equilibrium solution. Thus, the general solution to the ODE is

$$y(t) \;=\; ce^{0.08t} + 500,000.$$

Since $y(0) = 120,000$, we have $120,000 = c + 500,000$ or $c = -380,000$. Hence, the solution to the initial-value problem is

$$y(t) \;=\; 500,000 - 380,000e^{0.08t}.$$

To see if Tanya makes it through college, we determine when $y(t) = 0$. Solving

$$0 \;=\; 500,000 - 380,000e^{0.08t}$$

for t gives

$$t \;=\; \frac{\ln(25/19)}{0.08} \;\approx\; 3.43.$$

Sadly, Tanya does not have enough money to finish college in four years, as her money runs out after 3.43 years.