1. The simplest differential equation with $y(t) = 2t$ as a solution is $dy/dt = 2$. The initial condition $y(0) = 3$ specifies the desired solution.

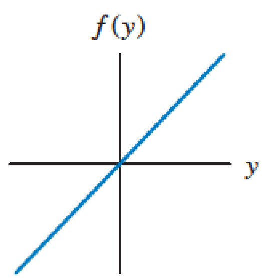
2. By guessing or separating variables, we know that the general solution is $y(t) = y_0e^{3t}$, where $y(0) = y_0$ is the initial condition.

4. Since the question only asks for one solution, look for the simplest first. Note that $y(t) = 0$ for all t is an equilibrium solution. There are other equilibrium solutions as well.

5. The right-hand side is zero for all t only if $y = -1$. Consequently, the function $y(t) = -1$ for all t is the only equilibrium solution.

7. The equations $dy/dt = y$ and $dy/dt = 0$ are first-order, autonomous, separable, linear, and homogeneous.

9. The graph of $f(y)$ must cross the y-axis from negative to positive at $y = 0$. For example, the graph of the function $f(y) = y$ produces this phase line.



11. True. We have $dy/dt = e^{-t}$, which agrees with $|y(t)|$.

17. True. Note that the function $y(t) = 3$ for all t is an equilibrium solution for the equation. The Uniqueness Theorem says that graphs of different solutions cannot touch. Hence, a solution with $y(0) > 3$ must have $y(t) > 3$ for all t.

19. False. By the Uniqueness Theorem, graphs of different solutions cannot touch. Hence, if one solution $y_1(t) \to \infty$ as t increases, any solution $y_2(t)$ with $y_2(0) > y_1(0)$ satisfies $y_2(t) > y_1(t)$ for all t. Therefore, $y_2(t) \to \infty$ as t increases.

20. False. The general solution of this differential equation has the form $y(t) = ke^t + \alpha e^{-t}$, where k is any constant and α is a particular constant (in fact, $\alpha = -1/2$). Choosing $k = 0$, we obtain a solution that tends to 0 as $t \to \infty$.

21. **(a)** The equation is autonomous, separable, and linear and nonhomogeneous.

(b) The general solution to the associated homogeneous equation is $y_h(t) = ke^{-2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha$. Then

$$\frac{dy_p}{dt} + 2y_p = 2\alpha.$$

Consequently, we must have $2\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3/2$, and the general solution to the nonhomogeneous equation is

$$y(t) = \tfrac{3}{2} + ke^{-2t}.$$

23. **(a)** The equation is linear and nonhomogeneous. (It is nonautonomous as well.)

(b) The general solution of the associated homogeneous equation is $y_h(t) = ke^{3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{7t}$. Then

$$\frac{dy_p}{dt} - 3y_p = 7\alpha e^{7t} - 3\alpha e^{7t} = 4\alpha e^{7t}.$$

Consequently, we must have $4\alpha = 1$ for $y_p(t)$ to be a solution. Hence, $\alpha = 1/4$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{3t} + \tfrac{1}{4}e^{7t}.$$

25. **(a)** This equation is linear and nonhomogeneous.

(b) To find the general solution, we first note that $y_h(t) = ke^{-5t}$ is the general solution of the associated homogeneous equation.

To get a particular solution of the nonhomogeneous equation, we guess

$$y_p(t) = \alpha \cos 3t + \beta \sin 3t.$$

Substituting this guess into the nonhomogeneous equation gives

$$\begin{aligned}\frac{dy_p}{dt} + 5y_p &= -3\alpha \sin 3t + 3\beta \cos 3t + 5\alpha \cos 3t + 5\beta \sin 3t \\ &= (5\alpha + 3\beta)\cos 3t + (5\beta - 3\alpha)\sin 3t.\end{aligned}$$

In order for $y_p(t)$ to be a solution, we must solve the simultaneous equations

$$\begin{cases} 5\alpha + 3\beta = 0 \\ 5\beta - 3\alpha = 1. \end{cases}$$

From these equations, we get $\alpha = -3/34$ and $\beta = 5/34$. Hence, the general solution is

$$y(t) = ke^{-5t} - \tfrac{3}{34}\cos 3t + \tfrac{5}{34}\sin 3t.$$

26. **(a)** This equation is linear and nonhomogeneous.

(b) We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{2y}{1+t} = t$$

and note that the integrating factor is

$$\mu(t) = e^{\int -2/(1+t)\,dt} = e^{-2\ln(1+t)} = \frac{1}{(1+t)^2}.$$

Multiplying both sides of the differential equation by $\mu(t)$, we obtain

$$\frac{1}{(1+t)^2}\frac{dy}{dt} - \frac{2y}{(1+t)^3} = \frac{t}{(1+t)^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{(1+t)^2}\right) = \frac{t}{(1+t)^2}.$$

Integrating both sides with respect to t and using the substitution $u = 1 + t$ on the right-hand side, we obtain

$$\frac{y}{(1+t)^2} = \frac{1}{1+t} + \ln|1+t| + k,$$

where k can be any real number. The general solution is

$$y(t) = (1+t) + (1+t)^2 \ln|1+t| + k(1+t)^2.$$

29. **(a)** This equation is linear and nonhomogeneous.

(b) First we note that the general solution of the associated homogeneous equation is ke^{-3t}.

Next we use the technique suggested in Exercise 19 of Section 1.8. We could find particular solutions of the two nonhomogeneous equations

$$\frac{dy}{dt} = -3y + e^{-2t} \qquad \text{and} \qquad \frac{dy}{dt} = -3y + t^2$$

separately and add the results to obtain a particular solution for the original equation. However, these two steps can be combined by making a more complicated guess for the particular solution.

We guess $y_p(t) = ae^{-2t} + bt^2 + ct + d$, and we have

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2ae^{-2t} + 2bt + c + 3ae^{-2t} + 3bt^2 + 3ct + d \\ &= ae^{-2t} + 3bt^2 + (2b + 3c)t + (c + 3d).\end{aligned}$$

Hence, for $y_p(t)$ to be a solution we must have $a = 1, b = \frac{1}{3}, c = -\frac{2}{9}$, and $d = \frac{2}{27}$. Therefore, a particular solution is $y_p(t) = e^{-2t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$. and the general solution is

$$y(t) = ke^{-3t} + e^{-2t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}.$$

31. **(a)** This equation is linear and nonhomogeneous. (It is nonautonomous as well.)

(b) The general solution of the associated homogeneous equaion is $y_h(t) = ke^{2t}$. To find a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha \cos 4t + \beta \sin 4t$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -4\alpha \sin 4t + 4\beta \cos 4t - 2(\alpha \cos 4t + \beta \sin 4t) \\ &= (-2\alpha + 4\beta) \cos 4t + (-4\alpha - 2\beta) \sin 4t.\end{aligned}$$

Consequently, we must have

$$(-2\alpha + 4\beta) \cos 4t + (-4\alpha - 2\beta) \sin 4t = \cos 4t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} -2\alpha + 4\beta = 1 \\ -4\alpha - 2\beta = 0. \end{cases}$$

Hence, $\alpha = -1/10$ and $\beta = 1/5$, and the general solution of the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{1}{10} \cos 4t + \tfrac{1}{5} \sin 4t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k - \tfrac{1}{10}.$$

Since the initial condition is $y(0) = 1$, we see that $k = 11/10$. The desired solution is

$$y(t) = \tfrac{11}{10} e^{2t} - \tfrac{1}{10} \cos 4t + \tfrac{1}{5} \sin 4t.$$

33. **(a)** The equation is separable because

$$\frac{dy}{dt} = (t^2 + 1)y^3.$$

(b) Separating variables and integrating, we have

$$\begin{aligned} \int y^{-3}\,dy &= \int (t^2+1)\,dt \\ \frac{y^{-2}}{-2} &= \frac{t^3}{3} + t + c \\ y^{-2} &= -\frac{2}{3}t^3 - 2t + k. \end{aligned}$$

Using the initial condition $y(0) = -1/2$, we get that $k = 4$. Therefore,

$$y^2 = \frac{1}{4 - 2t - \frac{2}{3}t^3}.$$

Taking the square root of both sides yields

$$y = \frac{\pm 1}{\sqrt{4 - 2t - \frac{2}{3}t^3}}.$$

In this case, we take the negative square root because $y(0) = -1/2$. The solution to the initial-value problem is

$$y(t) = \frac{-1}{\sqrt{4 - 2t - \frac{2}{3}t^3}}.$$

34. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-5t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{-5t}$ rather than αe^{-5t} because αe^{-5t} is a solution of the homogeneous equation. Then

$$\begin{aligned} \frac{dy_p}{dt} + 5y_p &= \alpha e^{-5t} - 5\alpha te^{-5t} + 5\alpha te^{-5t} \\ &= \alpha e^{-5t}. \end{aligned}$$

Consequently, we must have $\alpha = 3$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-5t} + 3te^{-5t}.$$

Note that $y(0) = k$, so the solution to the initial-value problem is

$$y(t) = -2e^{-5t} + 3te^{-5t} = (3t - 2)e^{-5t}.$$

35. **(a)** This equation is linear and nonhomogeneous. (It is nonautonomous as well.)

(b) We rewrite the equation as

$$\frac{dy}{dt} - 2ty = 3te^{t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{-t^2}\frac{dy}{dt} - 2te^{-t^2}y = 3t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(e^{-t^2}y\right) = 3t,$$

and integrating both sides with respect to t, we obtain $e^{-t^2}y = \frac{3}{2}t^2 + k$, where k is an arbitrary constant. The general solution is

$$y(t) = \left(\tfrac{3}{2}t^2 + k\right)e^{t^2}.$$

To find the solution that satisfies the initial condition $y(0) = 1$, we evaluate the general solution at $t = 0$ and obtain $k = 1$. The desired solution is

$$y(t) = \left(\tfrac{3}{2}t^2 + 1\right)e^{t^2}.$$

37. **(a)** This equation is separable.

(b) We separate variables and integrate to obtain

$$\int \frac{1}{y^2}\,dy = \int (2t + 3t^2)\,dt$$

$$-\frac{1}{y} = t^2 + t^3 + k$$

$$y = \frac{-1}{t^2 + t^3 + k}.$$

To find the solution of the initial-value problem, we evaluate the general solution at $t = 1$ and obtain

$$y(1) = \frac{-1}{2 + k}.$$

Since the initial condition is $y(1) = -1$, we see that $k = -1$. The solution to the initial-value problem is

$$y(t) = \frac{1}{1 - t^2 - t^3}.$$

39. **(a)** The differential equation is separable.

(b) We can write the equation in the form

$$\frac{dy}{dt} = \frac{t^2}{y(t^3 + 1)}$$

and separate variables to get

$$\int y\,dy = \int \frac{t^2}{t^3 + 1}\,dt$$

$$\frac{y^2}{2} = \frac{1}{3}\ln|t^3 + 1| + c,$$

where c is a constant. Hence,

$$y^2 = \frac{2}{3}\ln|t^3 + 1| + 2c.$$

The initial condition $y(0) = -2$ implies

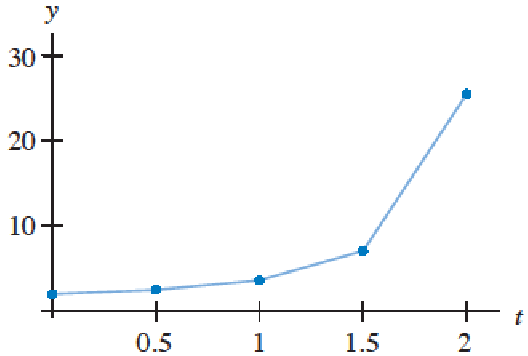
$$(-2)^2 = \frac{2}{3}\ln|1| + 2c.$$

Thus, $c = 2$, and

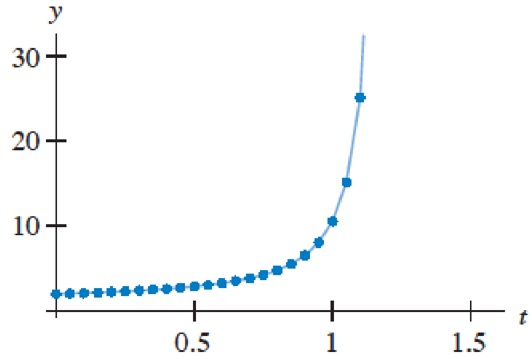
$$y(t) = -\sqrt{\tfrac{2}{3}\ln|t^3 + 1| + 4}.$$

We choose the negative square root because $y(0)$ is negative.

40. **(a)**



(b)



(c) Note that

$$\frac{dy}{dt} = (y-1)^2.$$

Separating variables and integrating, we get

$$\int \frac{1}{(y-1)^2}\,dy = \int 1\,dt$$

$$\frac{1}{1-y} = t + k.$$

From the intial condition, we see that $k = -1$, and we have

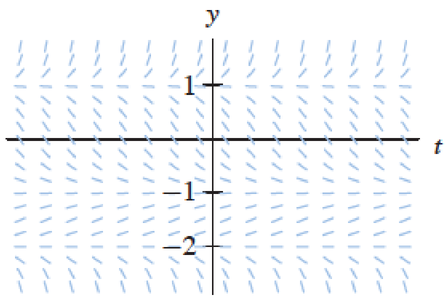
$$\frac{1}{1-y} = t - 1.$$

Solving for y yields

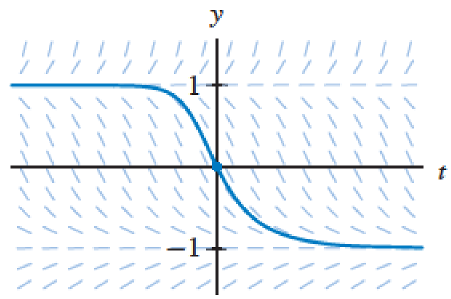
$$y(t) = \frac{t-2}{t-1},$$

which blows up as $t \to 1$ from below.

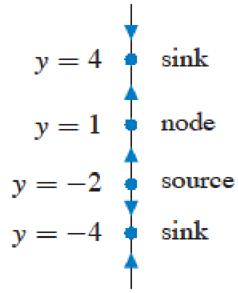
41. **(a)**



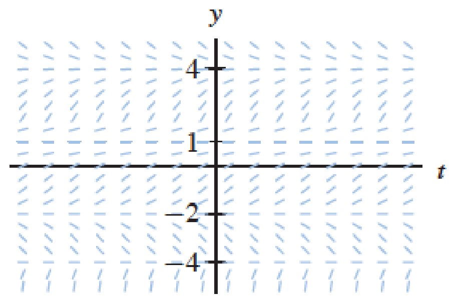
(b)



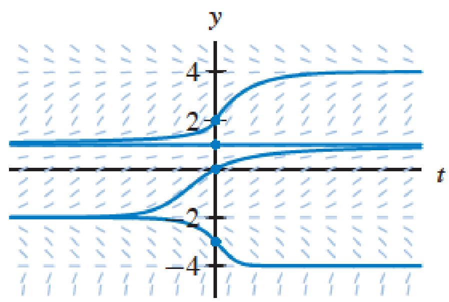
42. **(a)**



(b)



(c)



46. **(a)** Note that there is an equilibrium solution of the form $y = -1/2$.
Separating variables and integrating, we obtain

$$\int \frac{1}{2y+1}\,dy = \int \frac{1}{t}\,dt$$

$$\tfrac{1}{2}\ln|2y+1| = \ln|t| + c$$

$$\ln|2y+1| = (\ln t^2) + c$$

$$|2y+1| = c_1 t^2,$$

where $c_1 = e^c$. We can eliminate the absolute value signs by allowing the constant to be either positive or negative. In other words, $2y + 1 = k_1 t^2$, where $k_1 = \pm c_1$. Hence

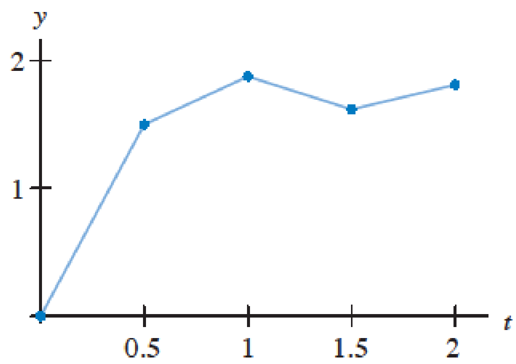
$$y(t) = kt^2 - \tfrac{1}{2},$$

where $k = k_1/2$.

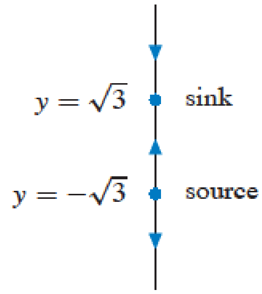
(b) As t approaches zero all the solutions approach $-1/2$. In fact, $y(0) = -1/2$ for every value of k.

(c) This example does not violate the Uniqueness Theorem because the differential equation is not defined at $t = 0$. So functions $y(t)$ can only be said to be solutions for $t \neq 0$.

47. **(a)** Using Euler's method, we obtain the values $y_0 = 0$, $y_1 = 1.5$, $y_2 = 1.875$, $y_3 = 1.617$, and $y_4 = 1.810$ (rounded to three decimal places).



(b)



(c) The phase line tells us that the solution with initial condition $y(0) = 0$ must be increasing. Moreover, its graph is below and asymptotic to the line $y = \sqrt{3}$ as $t \to \infty$. The oscillations obtained using Euler's method come from numerical error.

48. **(a)** If we let k denote the proportionality constant in Newton's law of cooling, the initial-value problem satisfied by the temperature T of the soup is

$$\frac{dT}{dt} = k(T - 70), \qquad T(0) = 150.$$

(b) We can solve the initial-value problem in part (a) using the fact that this equation is a nonhomogeneous linear equation. The function $T(t) = 70$ for all t is clearly an equilibrium solution to the equation. Therefore, the Extended Linearity Principle tells us that the general solution is

$$T(t) = 70 + ce^{kt},$$

where c is a constant determined by the initial condition. Since $T(0) = 150$, we have $c = 80$.

To determine k, we use the fact that $T(1) = 140$. We get

$$\begin{aligned} 140 &= 70 + 80e^k \\ 70 &= 80e^k \\ \tfrac{7}{8} &= e^k. \end{aligned}$$

We conclude that $k = \ln(7/8)$.

In order to find t so that the temperature is $100°$, we solve

$$100 = 70 + 80e^{\ln(7/8)t}$$

for t. We get $\ln(3/8) = \ln(7/8)t$, which yields $t = \ln(3/8)/\ln(7/8) \approx 7.3$ minutes.

49. **(a)** Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation depends only on t. The only choices are equations (i) and (iv). Because the slopes are negative for $t > 1$ and positive for $t < 1$, this slope field corresponds to equation (iv).

(b) This slope field has an equilibrium solution corresponding to the line $y = 1$, as does equations (ii), (v), (vii), and (viii). Equations (ii), (v), and (viii) are autonomous, and this slope field is not constant along horizontal lines. Consequently, it corresponds to equation (vii).

(c) This slope field is constant along horizontal lines, so it corresponds to an autonomous equation. The autonomous equations are (ii), (v), and (viii). This field does not correspond to equation (v) because it has the equilibrium solution $y = -1$. The slopes are negative between $y = -1$ and $y = 1$. Consequently, this field corresponds to equation (viii).

(d) This slope field depends both on y and on t, so it can only correspond to equations (iii), (vi), or (vii). It does not correspond to (vii) because it does not have an equilibrium solution at $y = 1$. Also, the slopes are positive if $y > 0$. Therefore, it must correspond to equation (vi).

50. **(a)** Let t be time measured in years with $t = 0$ corresponding to the time of the first deposit, and let $M(t)$ be Beth's balance at time t. The 52 weekly deposits of \$20 are approximately the same as a continuous yearly rate of \$1,040. Therefore, the initial-value problem that models the growth in savings is

$$\frac{dM}{dt} = 0.011M + 1{,}040, \qquad M(0) = 400.$$

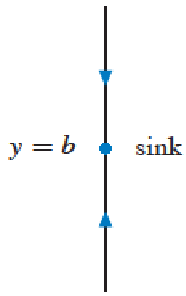
(b) The differential equation is both linear and separable, so we can solve the initial-value problem by separating variables, using an integrating factor, or using the Extended Linearity Principle. We use the Extended Linearity Principle.

The general solution of the associated homogeneous equation is $ke^{0.011t}$. We obtain one particular solution of the nonhomogeneous equation by determining its equilibrium solution. The equilibrium point is $M = -1{,}040/0.011 \approx -94{,}545$. Therefore, the general solution of the nonhomogeneous equation is

$$M(t) = ke^{0.011t} - 94{,}545.$$

Since $M(0) = 400$, we have $k = 94{,}945$, and after four years, Beth balance is $M(4) \approx 94{,}945e^{0.044} - 94{,}545 \approx \$4{,}671$.

51. **(a)**



(b) As $t \to \infty$, $y(t) \to b$ for every solution $y(t)$.

(c) The equation is separable and linear. Hence, you can find the general solution by separating variables or by either of the methods for solving linear equations (undetermined coefficients or integrating factors).

(d) The associated homogeneous equation is $dy/dt = -(1/a)y$, and its general solution is $ke^{-t/a}$. One particular solution of the nonhomogeneous equation is the equilibrium solution $y(t) = b$ for all t. Therefore, the general solution of the nonhomogeneous equation is

$$y(t) = ke^{-t/a} + b.$$

(e) The authors love all the methods, just in different ways and for different reasons.

(f) Since $a > 0$, $e^{-t/a} \to 0$ as $t \to \infty$. Hence, $y(t) \to b$ as $t \to \infty$ independent of k.

52. **(a)** The equation is separable. Separating variables and integrating, we obtain

$$\int y^{-2}\, dy = \int -2t\, dt$$
$$-y^{-1} = -t^2 + c,$$

where c is a constant of integration. Multiplying both sides by -1 and inverting yields

$$y(t) = \frac{1}{t^2 + k},$$

where k can be any constant. In addition, the equilibrium solution $y(t) = 0$ for all t is a solution.

(b) If $y(-1) = y_0$, we have

$$y_0 = y(-1) = \frac{1}{1+k}$$

so

$$k = \frac{1}{y_0} - 1.$$

As long as $k > 0$, the denominator is positive for all t, and the solution is bounded for all t. Hence, for $0 \le y_0 < 1$, the solution is bounded for all t. (Note that $y_0 = 0$ corresponds to the equilibrium solution.) All other solutions escape to $\pm\infty$ in finite time.