ODE Math 304, Fall 2006

Lab Project #4

Solving the Two-Body Problem

DUE DATE: Friday Dec. 8, 5:00 pm.

The goal of this project is to "solve" the classical two-body problem from celestial mechanics. Although explicit analytic solutions are impossible to obtain, there are powerful geometric and qualitative techniques available which make the problem tractable. You will use many of the methods developed for nonlinear systems thus far, specifically ideas from Hamiltonian systems theory such as integrals of motion. Many of the arguments rely heavily on identities and concepts from multivariable calculus.

Although most of the project involves calculations you should do by hand, there are a few questions which require some form of technology. For these questions you may use the DETools CD-ROM software that came with the textbook, the ODE software by John Polking linked from the course webpage or Maple.

The last three classes of the course will be devoted to completing this lab project. It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be appropriately acknowledged. Your report should provide coherent answers to each of the following questions. It does not need to be typed. Only **one project per group** should be submitted.

The Kepler Problem

In 1609, Johannes Kepler, writing in his *Astronomia Nova*, presented his three famous laws of motion for a planet traveling around the sun. First, the planets travel on ellipses with the sun at one focus. Second, the area swept out over equal time intervals is always the same and third, the period squared divided by the length of the semi-major axis cubed is identical for all planets. Later in the 17th century, Newton basically invented calculus to verify Kepler's laws, using his famous inverse square law of gravity. This was a major triumph for mathematics and physics and marked the dawn of modern science and technology. In this project, you will prove Kepler's laws using the more modern techniques developed in the theory of ordinary differential equations.

Consider two celestial bodies with positions $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^3$ and masses m_1, m_2, respectively, interacting according to Newton's inverse square law of gravitation. Each body is attracted to the other in proportion to the product of the masses and the inverse square of the distance between the two bodies. Let d be the distance between the bodies, $d = ||\mathbf{q}_2 - \mathbf{q}_1||$, where $||\mathbf{v}||$ is the standard Euclidean length of the vector $\mathbf{v}$.

To write down the differential equation for the two-body problem, we must write the force on each body as a vector. The unit vector which points from $\mathbf{q}_1$ towards $\mathbf{q}_2$ is given by $(\mathbf{q}_2 - \mathbf{q}_1)/d$, while the unit vector pointing from $\mathbf{q}_2$ towards $\mathbf{q}_1$ is the opposite of this, $(\mathbf{q}_1 - \mathbf{q}_2)/d$. (Recall that a unit vector is obtained when dividing a given vector by its length.) We then multiply the magnitude of the force given by Newton's law of gravitation by each unit vector. Choosing units so that the gravitational constant is $G = 1$ (the proportionality constant), the system of differential equations for the two-body problem (using $F = ma$) is given by

$$m_1 \ddot{\mathbf{q}}_1 = \frac{m_1 m_2 (\mathbf{q}_2 - \mathbf{q}_1)}{d^3} \tag{1}$$

$$m_2 \ddot{\mathbf{q}}_2 = \frac{m_1 m_2 (\mathbf{q}_1 - \mathbf{q}_2)}{d^3} \tag{2}$$

Since each body $\mathbf{q}_i$ has three components, this is really a system of 6 second-order differential equations. Changing to a first-order system with the velocities of each component as variables leads to a 12-dimensional system! "Solving" the 2-body problem essentially means answering the following question: Given the initial position and initial velocity of each body (12 numbers), what is the ensuing motion of the two bodies? Although this seems difficult (we barely understand two-dimensional phase portraits!) there are several useful reductions that can be performed which greatly simplify the problem.

Project Exercises

1. The **center of mass** of the system $\bar{\mathbf{q}}(t)$ is defined as

$$\bar{\mathbf{q}}(t) = \frac{m_1 \mathbf{q}_1(t) + m_2 \mathbf{q}_2(t)}{m_1 + m_2}.$$

As the bodies move, so does the center of mass. Show that $\bar{\mathbf{q}}(t) = \mathbf{c}_1 t + \mathbf{c}_2$ for some constant vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. (*Hint:* Differentiate $\bar{\mathbf{q}}(t)$ twice.) What are the values of $\mathbf{c}_1$ and $\mathbf{c}_2$ in terms of the initial conditions $\mathbf{q}_1(0), \mathbf{q}_2(0), \dot{\mathbf{q}}_1(0), \dot{\mathbf{q}}_2(0)$? This shows that the center of mass is traveling along a line in space at a constant velocity. The constants $\mathbf{c}_1$ and $\mathbf{c}_2$ are **constants of motion** (integrals) since they are fixed once the initial conditions are specified.

2. The center of mass calculation above actually cuts the dimension of the problem in half from 12 to 6. To see this, we change from the $\mathbf{q}_1, \mathbf{q}_2$ coordinate system to a $\mathbf{Q}_1, \mathbf{Q}_2$ coordinate system which has the center of mass fixed at the origin. Let $\mathbf{Q}_1 = \mathbf{q}_1 - \bar{\mathbf{q}}$ and $\mathbf{Q}_2 = \mathbf{q}_2 - \bar{\mathbf{q}}$ be our new position coordinates. (Remember, these coordinates are really functions of time.)

   a. Show that $m_1 \mathbf{Q}_1(t) + m_2 \mathbf{Q}_2(t) = 0$. Thus, the center of mass of the new system is always at the origin.

   b. Find the new equations of motion in $\mathbf{Q}_1, \mathbf{Q}_2$ coordinates. *Hint:* Substitute $\mathbf{q}_1 = \mathbf{Q}_1 + \bar{\mathbf{q}}$ and $\mathbf{q}_2 = \mathbf{Q}_2 + \bar{\mathbf{q}}$ into equations (1) and (2).

   c. By using the identity $m_1 \mathbf{Q}_1(t) + m_2 \mathbf{Q}_2(t) = 0$, show that the new equations of motion decouple into

$$\ddot{\mathbf{Q}}_1 = \frac{-m_2^3}{(m_1 + m_2)^2} \frac{\mathbf{Q}_1}{||\mathbf{Q}_1||^3} \tag{3}$$

$$\ddot{\mathbf{Q}}_2 = \frac{-m_1^3}{(m_1 + m_2)^2} \frac{\mathbf{Q}_2}{||\mathbf{Q}_2||^3} \tag{4}$$

   It follows that we only need to solve one of these differential equations, say the one for $\mathbf{Q}_1$. Once we have $\mathbf{Q}_1(t)$, we then use $m_1 \mathbf{Q}_1(t) + m_2 \mathbf{Q}_2(t) = 0$ to obtain $\mathbf{Q}_2(t)$.

3. We have reduced the two-body problem down to the system

$$\ddot{\mathbf{q}} = -\frac{\lambda \mathbf{q}}{||\mathbf{q}||^3} \tag{5}$$

where $\mathbf{q} \in \mathbb{R}^3$ and $\lambda > 0$ is a constant depending on the masses. This problem is known as the **Kepler problem** and more generally, is an example of a **central force problem**. If $\mathbf{q}$ is the position of the Earth (or any planet, asteroid, satellite, etc.), then this ODE models the motion of the Earth around the sun (assumed to be fixed at the origin.) Although the problem is six-dimensional, there are further reductions available using conservation of angular momentum.

a. Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be arbitrary vectors in $\mathbb{R}^3$ depending on time t. Using standard rectangular coordinates and the formula for the cross product of two vectors $\mathbf{u} \times \mathbf{v}$, show that

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}}$$

This is the product rule for the cross product.

b. Using the identity just proven, show that

$$\mathbf{q}(t) \times \dot{\mathbf{q}}(t) = \Omega \quad \forall t \tag{6}$$

for some constant vector $\Omega \in \mathbb{R}^3$. The vector Ω is called the **angular momentum** and is another constant of motion.

c. Suppose that $\Omega \neq \mathbf{0}$. Show that $\mathbf{q}(t)$ is orthogonal to Ω for all time t. In other words, the motion of $\mathbf{q}$ lies in a fixed plane with normal vector Ω.

4. By making another change of coordinates, we can assume that the plane of motion for $\mathbf{q}$ is the xy-plane. In other words, choose coordinates so that the angular momentum Ω lies on the z-axis, $\Omega = [0, 0, \omega]$. We will not actually do this here but the change of coordinates is similar to that used for moving the center of mass to the origin. This reduces the problem from 6 dimensions to 4 (two position variables x and y and their respective velocities.) From now on, we think of $\mathbf{q}$ as a vector in the plane and set $\mathbf{q} = [x, y]$.

a. Show that with our new setup, identity (6) reduces to

$$x\dot{y} - y\dot{x} = \omega \tag{7}$$

b. Changing to polar coordinates for the moment, $x = r\cos\theta, y = r\sin\theta$, show that identity (7) becomes

$$\dot{\theta} = \frac{\omega}{r^2} \tag{8}$$

Consequently, if we can find $r(t)$, then integrating equation (8) once will yield $\theta(t)$. Also notice that $\dot{\theta}$ is strictly positive or negative, depending on the sign of ω. This means that $\theta(t)$ is monotonic and the motion is always in the same direction, counterclockwise or clockwise.

c. In polar coordinates, the area $A(t)$ swept out by a vector of radius $r(t)$ satisfies $\dot{A} = \frac{1}{2}r^2\dot{\theta}$. (Recall the formula for the area of a sector is $A = r^2\theta/2$.) Use this to verify **Kepler's Second Law**: The line segment joining the sun to a planet sweeps out equal areas in equal times. *Hint:* Show that $A(t + \Delta t) - A(t)$ depends only on Δt.

5. To fully understand the motion of the planet, we need to find an expression for the radius $r(t)$. There are two cases depending on whether $\Omega = \mathbf{0}$ or $\Omega \neq \mathbf{0}$. We will need to utilize the vector identity

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \;=\; (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{9}$$

where $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are any three vectors in $\mathbb{R}^3$ and $\mathbf{u} \cdot \mathbf{w}$ is the usual dot product.

a. Let $\mathbf{q}(t) = [x(t), y(t)]$ so that $||\mathbf{q}(t)|| = \sqrt{x^2(t) + y^2(t)}$. Show that

$$\frac{d}{dt}(||\mathbf{q}||) \;=\; \frac{\dot{\mathbf{q}} \cdot \mathbf{q}}{||\mathbf{q}||}\,. \tag{10}$$

b. Use the two identities (9) and (10) to prove that

$$\frac{d}{dt}\left(\frac{\mathbf{q}}{||\mathbf{q}||}\right) \;=\; \frac{\Omega \times \mathbf{q}}{||\mathbf{q}||^3} \tag{11}$$

6. **Angular Momentum** $\Omega = 0$

a. What relationship must the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ satisfy so that $\Omega = \mathbf{0}$?

b. Using equation (11) show that if $\Omega = \mathbf{0}$, then the unit vector $\mathbf{q}/||\mathbf{q}||$ is constant for all time. Conclude that the motion of the body is on a fixed line through the origin. Why does this make physical sense given your answer to part **a.**?

c. Since the motion is on a line, set $\mathbf{q}(t) = (x(t), 0)$ where $x(t) > 0$. Thus we restrict the motion to the positive x-axis. Recall that the sun is fixed at the origin. We say that a **collision** occurs at time T if

$$\lim_{t \to T^-} x(t) \;=\; 0$$

Note that the ODE (5) is undefined if $\mathbf{q} = \mathbf{0}$. Let $\dot{x} = v$ and convert the ODE (5) into a planar first-order system in the variables x and v.

d. Show that the resulting system is Hamiltonian and give a Hamiltonian function $H(x, v)$. The function H in mechanics is usually called the **energy**. Since solutions lie on level curves of H, energy is a conserved quantity.

e. Using your Hamiltonian and other techniques for drawing planar phase portraits, sketch the phase portrait in the xv-plane. You may assume that $x \geq 0$ and $\lambda = 1$. Feel free to use technology to help with your sketch.

f. Using the phase portrait, describe the fate of different solutions in forward time. Be sure to differentiate between the case $v(0) = v_0 \leq 0$ and $v(0) = v_0 > 0$. Do collisions always occur? How does this depend on the value of the energy H? EXTRA CREDIT: Prove that any collision will occur in finite time.

7. **Angular Momentum** $\Omega \neq 0$

a. Using equation (11) show that if $\Omega \neq \mathbf{0}$, then

$$\lambda\left(\frac{\mathbf{q}}{||\mathbf{q}||} + \mathbf{E}\right) \;=\; \dot{\mathbf{q}} \times \Omega \tag{12}$$

where $\mathbf{E}$ is a constant vector, our final integral of motion.

b. Take the dot product of both sides of equation (12) with $\mathbf{\Omega}$ to conclude that $\mathbf{E} \cdot \mathbf{\Omega} = 0$. Consequently, $\mathbf{E}$ is in the plane of motion.

c. Using the vector identity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, take the dot product of both sides of equation (12) with $\mathbf{q}$ (on the left) to conclude that

$$\mathbf{E} \cdot \mathbf{q} + ||\mathbf{q}|| = \frac{\omega^2}{\lambda}. \tag{13}$$

d. Again, there is a change of variables (a rotation of the plane of motion) so that $\mathbf{E}$ lies on the positive x-axis. Thus, set $\mathbf{E} = [e, 0]$ with $e \geq 0$. Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$r = \frac{\omega^2/\lambda}{1 + e\cos\theta}. \tag{14}$$

This gives an equation for $r(t)$ in terms of $\theta(t)$. Although we don't actually have an expression for $\theta(t)$, we know that it is a monotonic function. By substituting the equation for $r(t)$ into the identity $\dot{\theta} = \omega/r^2$ we have a first-order separable ODE in terms of the dependent variable θ

$$\frac{d\theta}{dt} = \frac{\lambda^2}{\omega^3}(1 + e\cos\theta)^2. \tag{15}$$

Unfortunately, this can not be solved using elementary functions. The problem is said to be **solvable up to quadrature**.

e. After all this work it is a tad depressing to find out the problem can not be solved with "nice" functions. However, the good news is that we can describe the motion of the planet determined by $\mathbf{q}$ completely. Equation (14) is the equation of a conic section in polar coordinates! This verifies **Kepler's First Law**, that the motion of the planet travels on an ellipse with the sun at one of the foci. To find the location of a planet at a specific time t, we need to numerically integrate equation (15) to find the exact angle $\theta(t)$.

The type of conic section depends on the value of the constant e. Show that if $e = 0$, the motion of the planet is circular. What is the period T of this solution, that is, what is the length of a year? What is the radius a of the circle? Verify **Kepler's Third Law** for the case $e = 0$:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\lambda}$$

Note that this ratio only depends on the constant λ, which in turn, depends on the gravitational constant and attractive force from the sun. Assuming this is the same in a given solar system, the ratio T^2/a^3 is thus identical for any planet!

f. Show that the conic section described by equation (14) is an ellipse if $0 < e < 1$, a parabola when $e = 1$ and a hyperbola for $e > 1$. To make things easier, you may assume that $\omega^2/\lambda = 1$. *Hint:* One approach is to graph the conic section and find an expression in Cartesian coordinates. It may help to recall the definitions of the various conic sections. To graph a conic section using Maple, use the command `polarplot` . You must first load the `plots` package by typing `with(plots);` . Typing

```
polarplot(1/(1+0.5*cos(theta)));
```

for example, will draw the conic section with $e = 1/2$. You might find it useful to sketch over a limited range for θ. For example,

```
polarplot(1/(1+0.5*cos(theta)),theta=0..Pi);
```

only evaluates the polar equation from $0 \leq \theta \leq \pi$.

g. EXTRA CREDIT: For the case $0 < e < 1$, let T be the period of the elliptical orbit and let a be the length of the semi-major axis of the ellipse. Verify **Kepler's Third Law** for elliptical orbits:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\lambda}$$

You will need to use the formula for the area of an ellipse in terms of eccentricity e and a.