ODE Math 304-01, Fall 2004

Computer Project #3

A Family of Predator-Prey Equations

DUE DATE: Friday Dec. 3, in class.

The goal of this project is to apply techniques for investigating planar nonlinear systems (linearization about equilibrium points, nullclines, bifurcation theory, etc.) to a population model of a predator-prey system. For this project you may use MAPLE or the ODE software by John Polking available at `http://math.rice.edu/~dfield/dfpp.html`. You may have to adjust your printing settings to print from the web using the Polking software. (See instructions on his website.) The command you will need for MAPLE is `phaseportrait` . Be sure to load the ODE package by typing `with(DEtools):`. You can learn about the command and see some special examples by typing `?phaseportrait` at a command prompt.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be appropriately acknowledged. Your report should be typed, providing coherent answers to each of the following questions. Only **one project per group** need be submitted.

A Model for a Predator-Prey System

Consider the following one-parameter family of nonlinear, first-order equations modeling a predator-prey system:

$$\begin{aligned} \frac{dx}{dt} &= 9x - bx^2 - 3xy \\ \frac{dy}{dt} &= -2y + xy \end{aligned}$$

where b is a parameter satisfying $0 \leq b \leq 9$. The variable x represents the population of a prey (say rabbits) while the variable y represents the population of predators that hunts and feeds on the prey (say foxes). How can you tell from the model that x corresponds to the prey while y corresponds to the predator?

The goal is to understand the interaction between the two species and determine the fate of each population as $t \to \infty$. This will depend on the parameter b so you should keep an eye out for any bifurcations. Keep in mind that these represent populations (units suppressed) so that we only want to consider the region where $x \geq 0$ and $y \geq 0$, that is, our phase portraits should only be drawn in the first quadrant including the two coordinate axes. "Negative" populations are physically unrealistic.

The Project

1. Suppose that $x_0 = x(0) = 0$, that is, the prey are extinct. Show that $x(t) = 0$ for all t, so that the prey remain extinct. (This makes good physical sense.) In this case, what is the fate of the predator population for different initial conditions of $y_0 = y(0)$? Does your answer depend on b? What type of model is being used for the predators when there are no prey? Give a physical interpretation of the model.

2. Suppose that $y_0 = y(0) = 0$, that is, the predators are extinct. Show that $y(t) = 0$ for all t, so that the predators remain extinct. (Again, this makes good physical sense.) In this case, what

is the fate of the prey population for different initial conditions of $x_0 = x(0)$? Does your answer depend on b? What type of model is being used for the prey when there are no predators? Give a physical interpretation of the model.

3. List all of the equilibrium points of the system. Your answer should depend on b. (Recall, $0 \leq b \leq 9$.) Given that $x \geq 0, y \geq 0$, how many equilibrium points are there? List any bifurcations you find, describing the numbers and locations of equilibria before, at and after the bifurcation. Is it possible for both populations to coexist?

4. Linearize the system about the equilibrium points you found in the previous question. Classify the type of each equilibrium point (saddle, source, spiral sink, etc.). Your answers will vary as b varies. Again, only consider those equilibria where $x \geq 0, y \geq 0$. Are there any new bifurcations that you didn't find in the previous question? Explain.

5. List and sketch the x- and y-nullclines. Since these may vary with b, you should draw several different pictures (before, at and after bifurcation values.) Sketch the direction field on the nullclines and in the region between nullclines. You may use software to help you check the direction field, but you should turn in hand-drawn sketches for this question.

6. Putting together the information obtained in the previous questions, sketch solutions in the phase plane ($x \geq 0, y \geq 0$) for various values of b, keeping in mind the bifurcation values you have located. For what values of b are the equilibrium points hyperbolic? In the hyperbolic cases, explain how the eigenvectors for the associated linear systems help you sketch the phase plane for the full nonlinear system. In the non-hyperbolic cases, does the linearization provide a useful approximation? Explain. **Note:** You may use MAPLE or the ODE software by John Polking to obtain phase portraits but be sure to explain how all the information obtained from previous questions fits into your picture.

7. Using all of the information obtained thus far, discuss the fate of both the predator and prey populations as b is varied from 0 to 9. List all the bifurcation values you have found and describe the long-term behavior of the populations before, at and after each bifurcation. Is it possible for both populations to *stably coexist*? (Not just coexist, but a small perturbation does not ruin this coexistence. One might say this is a robust as opposed to fragile ecosystem.)

8. The case $b = 0$ is particularly interesting because it is possible to find formulas for solution curves in the phase plane. The system is said to be **integrable** in this case. Let

$$L(x, y) \;=\; \frac{x}{3} + y - \frac{2}{3}\ln x - 3\ln y$$

be a two-variable function defined for $x > 0, y > 0$.

a. Let $(x(t), y(t))$ be a solution to the system of ODE's with $b = 0$ and initial condition (x_0, y_0). Show that if $L(x_0, y_0) = c$, then $L(x(t), y(t)) = c$ for all t. In other words, a solution on a level curve of $L(x, y)$ stays on the same level curve for all time. We call $L(x, y)$ an **integral** for the ODE.

b. Show that $(2, 3)$ is a minimum of $L(x, y)$. (Review your multi-variable calculus — good practice for next semester.) What does this tell us about the level curves near $(2, 3)$?

c. For the case $b = 0$, the system of ODE's has an equilibrium point (x^*, y^*) not on either coordinate axis. Is this equilibrium point hyperbolic? Using the fact that $L(x, y)$ is an integral, what type of solutions exist near (x^*, y^*) in the phase plane? Given the linearization about (x^*, y^*), why is it important to have the integral $L(x, y)$?