ODE Math 304-01, Fall 2004

Computer Project #2

RLC Circuits and Resonance

DUE DATE: Monday, Nov. 15, in class.

The goal of this project is for you to apply your knowledge of second-order linear ODE's to gain some insight into electrical circuits. In particular, you will use MAPLE to find analytic formulas of solutions as well as visualize solutions with graphs. You will learn about an important phenomenon called resonance which surfaces in many applications of second-order linear ODE's.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be appropriately acknowledged. Your report should provide coherent answers to each of the following questions. Only **one project per group** need be submitted.

The Equations for an RLC Circuit

An **RLC circuit** is one that contains each of the following:

- a voltage source $v_T(t)$ (time dependent)
- a **resistor** (something that dissipates electrical energy) with resistance $R \geq 0$
- an **inductor** (a device or component that stores magnetic energy) with inductance $L > 0$
- a **capacitor** (a device or component that stores electrical energy) with capacitance $C > 0$

It turns out, using **Kirchoff's current law**, the **Kirchoff voltage law** and **Ohm's law**, that the amount of current and voltage through the resistor, inductor and capacitor is completely determined by the current $i(t)$ flowing through the circuit and the voltage across the capacitor $v_C(t)$. We would like to know how these quantities change with time.

The equations for the voltage $v_C(t)$ and current $i(t)$ satisfy the nonautonomous linear system of differential equations

$$\begin{aligned}\frac{dv_C}{dt} &= \frac{i}{C}\\ \frac{di}{dt} &= -\frac{v_C}{L} - \frac{R}{L}i + \frac{v_T(t)}{L}\end{aligned}$$

This system is more commonly written as a second-order linear ODE with forcing:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_T(t) \tag{1}$$

which are the same equations as the forced harmonic oscillator. For this lab, we will always consider a periodic, sinusoidal voltage source ("forcing") $v_T(t) = a\sin(\omega t)$. For more information on the RLC circuit equations see Section 12.1 of the Hirsch, Smale, Devaney text or Lab 3.2 (pp. 362 - 364) of the Blanchard, Devaney, Hall text.

Some MAPLE Commands

All of the plots necessary for this lab can be generated using the basic MAPLE `plot` command. However, if you want to check your results, you can also use the `DEplot` command from Lab 1 to plot approximate numerical solutions to the second-order ODE. For example, suppose we wanted to find the solution to the initial-value problem

$$4\ddot{y} + 3\dot{y} + 10y \;=\; 3\cos(2t), \qquad y(0) = 5,\; y'(0) = -3$$

First, load the `DEtools` package by typing `with(DEtools);` Then, define the ODE by typing:

```
eqn:=4*diff(y(t),t$2)+3*diff(y(t),t)+10*y(t) = 3*cos(2*t);
```

and plot the solution for $t \in [0, 10]$ using

```
DEplot(eqn,y(t),t=0..10,[[y(0)=5,D(y)(0)=-3]],linecolor=blue,stepsize = 0.01);
```

As in Lab 1, adjusting the `linecolor` and `stepsize` options, the range of t-values, etc. will often be necessary to get reasonable, printable graphs.

Another useful command you may wish to run occasionally is `k := 'k';` which clears the value of k and allows you to treat k as a variable rather than a constant. This is useful if you have previously assigned a number to a variable but want to go back and use it as an unknown variable.

The Project

1. **An Example**

   Suppose we want to understand the voltage in an RLC circuit with $R = 1000$, $L = 2$, and $C = 10^{-6}$, and a voltage source $v_T = 20\sin(50t)$. We know the general solution to the ODE (1) looks like $v_C(t) = c_1 v_1(t) + c_2 v_2(t) + v_p(t)$, where $c_1 v_1(t) + c_2 v_2(t)$ is the general solution of the associated homogeneous equation and $v_p(t)$ is any particular solution.

   Begin by defining our constants:

   ```
   R := 1000:  L := 2:  C := 10^(-6):
   ```

   Note that the colon keeps the output from being returned on the screen, whereas a semi-colon returns the output of the command. To find $v_1(t), v_2(t)$ using MAPLE, we find the roots of the characteristic polynomial of the corresponding first order system:

   $$LC\lambda^2 + RC\lambda + 1 \;=\; 0.$$

   You can use the `fsolve` command to find these roots numerically:

   ```
   fsolve(L*C*lambda^2 + R*C*lambda + 1 = 0, lambda, complex);
   ```

   Call the roots $\alpha \pm i\beta$ and define them in a command line using `alpha :=` and `beta :=` . Now, the general solution of the homogeneous equation has the form $c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$. Define these two solutions here as functions using:

   ```
   v1 := t -> exp(alpha*t)*cos(beta*t);
   v2 := t -> exp(alpha*t)*sin(beta*t);
   ```

Next we need to find a particular solution v_p. Using the complexification method discussed in class, (this is also known as the **method of undetermined coefficients**,) guess a solution of the form $v_p(t) = ke^{i\,50t}$. Plug $v_p(t)$ into the ODE and find the value of the complex constant k. You should obtain the equation

$$-2500LCk + RC50ik + k \;=\; 20.$$

This can be solved with

```
fsolve(L*C*(-2500)*k + R*C*50*I*k + k = 20,k,complex);
```

Note that in MAPLE, the imaginary number i is typed as `I` . Define your solution to be the constant `k :=` . You can find v_p by taking the imaginary part of $ke^{i\,50t}$. Using MAPLE, this can be accomplished with the commands

```
assume(t,real);
Im(k*exp(I*50*t));
```

The `assume` command is used to tell MAPLE that t is a real variable. Copy and paste your solution to the function $v_p(t)$, that is, `vp := t ->` `...` . Then, define the general solution using

```
vc := t -> c1*v1(t) + c2*v2(t) + vp(t);
```

MAPLE is also useful for solving initial-value problems. To find the values of c_1 and c_2 that give a solution with $v(0) = 0$, $v'(0) = 1$, you can type

```
initeqs:={vc(0)=0,subs(t=0,diff(vc(t),t))=1}:
fsolve(initeqs,{c1,c2});
```

Defining the constants `c1 := ...` and `c2 := ...` give us the solution $v_C(t)$ to the initial-value problem.

Plot the solution $v_c(t)$ using the `plot` command. Be sure to choose an appropriate plot domain. Also plot the homogeneous solution $c_1v_1(t)+c_2v_2(t)$ to see why the solution v_C looks indistinguishable from a sinusoidal function. The homogeneous solution is usually referred to as the **transient part** of the solution while the particular solution $v_p(t)$ is often called the **steady-state solution**. It is the steady-state solution which determines the eventual behavior of the system while the transient part determines how quickly the solution approaches the steady-state.

Note: All that needs to be handed in for this problem is the formula for the solution to the initial-value problem, a good plot of this solution and a good plot of its homogeneous solution.

2. **Amplitude of the Steady-State Solution**

Consider the RLC circuit ODE

$$LC\ddot{v_C} + RC\dot{v_C} + v_C \;=\; a\sin(\omega t) \tag{2}$$

with periodic voltage source. How does the amplitude of the steady-state solution depend on the parameters L, R, C, a, ω? The following questions are to be done **without** MAPLE.

a. Find, by hand, the eigenvalues of the characteristic polynomial for the associated homogeneous equation. Show that if $R > 0$, then every solution of this equation tends to zero as $t \to \infty$. This is why the homogeneous solution if often described as "transient." (Recall that we assumed $L > 0, C > 0$ and $R \geq 0$.)

b. Show that a function of the form $v_p(t) = \alpha \cos(\omega t) - \beta \sin(\omega t)$ can be written in the form

$$v_p(t) \;=\; A \cos(\omega t + \phi)$$

where $A = \sqrt{\alpha^2 + \beta^2}$. A is the **amplitude** and ϕ is the **phase** of v_p. Note that $v_p(t)$ is equivalent to the real part of $(\alpha + i\beta)e^{i\omega t}$. Thus the amplitude of v_p is equivalent to the modulus of the complex coefficient $\alpha + i\beta$. (The modulus of a complex number $k = \alpha + i\beta$ is given by $|k| = \sqrt{\alpha^2 + \beta^2}$.)

c. Suppose that $R > 0$. Find, by hand, the complex formula for the steady-state solution. In other words, letting $v_p(t) = ke^{i\omega t}$, solve to find k. Then find $|k|$, the modulus of the complex coefficient, to obtain a formula for the amplitude of the steady-state solution. Your answer should depend on all five of our parameters.

d. How does the amplitude depend on a? Does this make physical sense?

e. How does the amplitude depend on R? Does this make physical sense?

f. Suppose that the frequency ω satisfies

$$\omega \;=\; \frac{1}{\sqrt{LC}}.$$

What happens to the amplitude as the resistance R approaches zero?

3. **Resonance**

The previous question hints at an important phenomenon called **resonance.** Suppose $L, C > 0$, but the resistance $R = 0$. (This is a very, very idealized situation, of course, even better than "superconductivity.") Let $p(\lambda)$ be the characteristic polynomial of the associated homogeneous equation for (2). There are two cases:

- $i\omega$ is not a root of $p(\lambda)$
- $i\omega$ is a root of $p(\lambda)$

a. Show that $i\omega$ is a root of $p(\lambda)$ if and only if

$$\omega \;=\; \frac{1}{\sqrt{LC}}.$$

This is the resonance case.

b. In the resonance case, find the formula for the general solution $v_C(t) = c_1 v_1(t) + c_2 v_2(t) + v_p(t)$ by hand. *Hint:* Guessing $v_p(t) = ke^{i\omega t}$ will fail. Amend this guess by considering what we did in the real case.

c. In the resonance case, as t gets larger, what function do solutions approach? What is the behavior of solutions as $t \to \infty$? What happens to the RLC circuit over time?

d. Give a "real-world" situation that might correspond to resonance. Is it a good or bad thing to have a resonant solution?

4. **An Example with Varying Frequency**

Using MAPLE, calculate the solutions to the RLC ODE (2) with $v_T(t) = a\sin(\omega t)$ and $R = 0, C = 10^{-6}, L = 1, a = 1, v_C(0) = 1, v'_C(0) = 0$, and $\omega = 900, 950, 1000, 1050, 1100$ (this is five different ODE's with five different solutions.) Plot each solution, making sure to chose your t interval carefully so as to obtain informative graphs. Note that it is necessary to actually compute the analytic solution in each case because the MAPLE command `DEplot` does not always give useful and/or accurate graphs of solutions. Does resonance occur for any of the ODE's? Explain.